A Review of the Measurement Problem in Quantum Mechanics

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Declaration

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Abstract

Quantum Mechanics has been a massively successful theory, but for all its success it is at odds with our experiences at the macroscopic or classical level. This tension between the quantum and classical worlds and the transition between them has been dubbed the Measurement Problem.

Here we review the Measurement Problem and describe it in the context of the double split experiment. We use this to define the problem and identify two major aspects, namely the Problem of Macro-objectification, the lack of superposition of macroscopic objects, and the Problem of Outcomes, the selection of a single outcome upon measurement of a superposition. Three more related, and more technical, problems are also identified.

We review in some detail the Environmental Decoherence and Dynamical Reduction programs, chosen as they are perceived to be the most realistic approaches to the Measurement problem. Environmental Decoherence occurs when the system of interest is allowed to weakly interact with the environment, and Dynamical Reduction programs are non-linear stochastic modifications made to the Schrodinger equation to cause reduction of the state to happen naturally.

We note that each introduces extra degrees of freedom external to the system of interest, the environment in the case of decoherence, and an external stochastic field in the case of Dynamical Reduction Programs. This commonality is then briefly demonstrated and discussed.

n' Oorsig van die Meetprobleem in Kwantummeganika

Ian Bennet Pittaway

Uittreksel

Kwantummeganika is 'n uiters suksesvolle teorie, maar ten spyte daarvan strook dit nie met ons ervaring op die makroskopiese of klassieke vlak nie. Hierdie spanning tussen die kwantum en klassieke wêrelde en die oorgang tussen hulle word na verwys as die meetprobleem.

Hier gee ons 'n oorsig van die meetprobleem in die konteks van die dubbel spleet eksperiment. Ons definieer aan die hand hiervan die probleem en identifiseer twee hoof aspekte, naamlik die probleem van makro-objektiwiteit, die afwesigheid van superposisie van makroskopiese voorwerpe, en die probleem van uitkomste, die seleksie van 'n enkele uitkoms by die meting van 'n superposisie. Drie verdere verwante, en meer tegniese, probleme is ook geidentifiseer.

'n Gedetaileerde oorsig word gegee van die omgewings dekoherensie en die dinamiese reduksie programme aangesien hulle algemeen as die mees realistiese benaderings tot die metings probleem gesien word. Omgewings dekoherensie kom voor wanneer 'n stelsel swak met die omgewing wisselwerk en die dinamiese reduksie program behels nie-lineêre stogastiese wysigings tot die Schrödinger vergelyking wat die natuurlike reduksie van die golffunksie teweeg bring.

Ons merk an dat beide programme die invoering van eksterne, bykomende vryheidsgrade behels, die omgewing in die geval van dekoherensie en 'n bykomende stogastiese veld in die geval van dinamiese reduksie. Hierdie gemeenskaplikheid word ook kortliks gedemonstreer en bespreek.

Contents

1. Introduction

Quantum mechanics, by virtue of its linear nature, describes the system as being able to be in several different states simultaneously, Schrödingers cat as being in a superposition of both dead and alive. Schrödinger hypothetically put the cat in the box with the intention of drawing attention to the absurdity of this statement, the inherent contradiction with what our classical sensibilities expect and what quantum mechanics suggest. For the case of quantum systems, of electrons and photons, no contradiction exists. Superpositions of such objects is a measured phenomenon, but then why, if cats are made of such things, do these principles not apply to them? Additionally, where along the scale from quantum to classical does this stop being the case?

The orthodoxy is to introduce a non-linear operation into the description that should be performed in the event of a measurement. This seemingly ad hoc and rather un-physical process, while perfectly consistent in its predictions, only draws attention to the contrast between the linear nature of the dynamics and the non-linear procedure of measurement. This discrepancy between these opposing views on the evolution of quantum dynamics has been part of its description since its conception nearly a century ago, even then being appreciated but hidden behind an artificial line separating what can be considered classical from quantum.

It is at this borderline where this tension is most apparent, during the process of measurement when typically quantum systems interact with classical measuring devices, giving the problem its name, the Measurement Problem. It is presented as a single problem, "why are the descriptions of quantum mechanics at odds with those of classical mechanics and relativity?", but it can be interpreted to include two distinct aspects, the lack of superposition of macroscopic objects and the identification of a single outcome upon measurement of a superimposed state, as well as various related issues.

As our technological ability increases and the edge of what can be considered "typically quantum" is pushed, most recently with interferometry experiments of masses consisting of up to 2,000 atoms [63], the quantum to classical transition can begin to be tested. If classical dynamics are to emerge from a quantum description, it is to happen somewhere within this

border realm, and it is here that we might find if our current description should be modified.

In the meantime, the orthodox approach to quantum mechanics has achieved unprecedented success in its predictive ability and is yet to be contradicted by any measurement outcome. For all this success in telling us what we find when we look, it offers nothing as far as what is actually there. As a result, many have attempted to resolve the measurement problem by reinterpreting standard quantum mechanics. This has resulted in debates that date back to the theory's founding and rage still to this day. Many adhere strictly to the standard dynamics, some only changing the semantics of quantum mechanics. This has resulted in a debate that is as much philosophical as it is over any physical problem.

To this end we have elected not to investigate the multitude of interpretations that exist outside of a brief review, restricting ourselves to the Environmental Decoherence scheme and Dynamical reduction models. These have been chosen for their particular approaches to the measurement problem or the role they play in the quantum to classical transition.

We will not pose any attempt at trying to resolve these issues here, greater minds have tried. Instead, in this thesis, we shall investigate the measurement process and the quantum-to-classical transition. We shall begin in section 2 by describing the problem in the context of the double split experiment, a common means of doing so. We shall then clearly define the measurement problem and its related aspects as we understand them, and discuss physical properties of the dynamics, namely unitarity or linearity, that would have to be changed in order to account for the measurement problem.

We shall then review the Environmental Decoherence in section 3. This interpretation agnostic scheme describes the suppression of interference when a system is allowed to weakly interact with its environment. There is a common misconception in the community that decoherence resolves the measurement problem [110, 114]. While this is not true, it does seemingly play a rather fundamental role in the transition from quantum to classical dynamics and offers an explanation for the apparent preferred basis with which we experience the world. Its extension into describing the environment not merely as a bath, but also a means of information propagation is also explored in section 3.4. There it is shown that this results in the phenomenon that has been referred to as Quantum Darwinism, the successful propagation of information that survives interaction with the environment.

In section 4 an investigation into the current state of Dynamical reduction models will be made. These are modified models of quantum mechanics that add stochastic terms to the Schrödinger equation that cause the system to undergo reduction naturally. We shall specifically look at the three most common models, the Ghirardi, Rimini, and Weber (GRW) model, the Continuous Spontaneous Localisation (CSL) model, and the quantum mechanics with universal position localization (QMUPL) model. Of the more popular attempts at resolving the measurement problem, reduction models, or alternatively, collapse models, offer one of the only mechanisms that might be experimentally falsified. As such, this description of the process of reduction, and its application to measurement, is the only interpretation investigated here.

In the final section, section 5, we shall discuss the overlap between these two investigated subjects in the context of the measurement problem and the way each invokes the idea of something beyond the system itself to explain their respective phenomenon. We shall very briefly and broadly review some other interpretations of quantum mechanics and discuss how each invokes a similar idea of extra degrees of freedom. Finally, in section 5.4 we shall briefly summarize the subjects we investigated and offer some concluding remarks.

2. The Problem of Measurement

The measurement problem, or macro-objectification problem, is intimately tied to the fundamental nature of quantum mechanics as a description of reality. One can forgo the view of quantum mechanics as a description of reality and use it as merely a mathematical tool for very accurately predicting the outcome of an experiment. In this case, there would be no measurement problem as standard quantum mechanics as a predictive tool has become one of the most accurate to date.

Here we shall assume that quantum mechanics in some way guides our interpretation of reality via its descriptions of the dynamics of a system of interest. In taking this view, we are immediately faced with the contradiction within the fundamental formulation of the dynamics described by standard quantum mechanics, i.e. the problem of measurement.

2.1 Postulates of Quantum Mechanical Dynamics

In describing the measurement problem, one is also required to understand the standard dynamics of quantum systems. To this end, we will begin with a brief review of the postulates of quantum mechanics. We shall separate the moment in which one might "measure" a "property" of the system from these postulates as it is there that the problem begins. We place these terms in quotation marks to emphasise their vagueness in the context of quantum mechanics, as we will discuss below. Regardless of one's philosophy on the interpretation of quantum mechanics, it is effectively universally accepted that the dynamics of a quantum system are governed by the standard postulates, as such we start with them.

Standard quantum mechanical dynamics, as is taught in any entry level course on quantum mechanics, is governed by the following rules:

1. A quantum system S is described by a normalized vector $|\phi\rangle$ belonging to some Hilbert space $\mathcal{H}_S$ which contains all information about that system's physical state. As such the system's vector is referred to as its statevector, or just its state.

   In the case one has only partial information about which of a list of N possible states

$\{|\psi_i\rangle\}$ the system might be in, one might choose to make use of the trace one positive semi–definite density matrix ρ as a description of the system,

$$\rho = \sum_i^N p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| , \tag{2.1.1}$$

where p_i is the probability of state $|\psi_i\rangle$ being the state of the system. Note that this implies $\mathrm{Tr}\,\rho = 1$. This then describes a statistical mixed state, or just a mixed state, which can be identified by the property $\rho^2 \neq \rho$. This is in contrast to a pure state, where $\rho^2 = \rho$, in which the state of the system is known completely and the density matrix reduces to a projection operator $\rho = |\psi\rangle \langle\psi|$.

2. A property O that one might attempt to assign to a system is referred to as an observable. It is represented by a Hermitian operator $\hat{O}$ that acts on the Hilbert space $\mathcal{H}_S$ of the system whose property one wishes to know about. The possible values that the system's property might take is then given by the eigenvalues o_n of the corresponding observable's operator $\hat{O}$, with associated eigenvector given by,

$$\hat{O}\left|o_n\right\rangle = o_n \left|o_n\right\rangle , \tag{2.1.2}$$

where we have assumed a discrete and non-degenerate spectrum of the observable for simplicity as further details are not required for an understanding of the measurement problem. The Hermiticity of the observable operator $\hat{O}$ then ensures that the eigenvectors $|o_n\rangle$ form a orthonormal basis for Hilbert space $\mathcal{H}_S$ with real valued eigenvalues o_n.

3. The non-relativistic dynamics of a quantum state can be described by the Schrödinger equation,

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi\right\rangle = \hat{H}\left|\psi\right\rangle , \tag{2.1.3}$$

where the Hermitian operator $\hat{H}$ is the system's Hamiltonian. The dynamics of the density matrix is given by

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\rho = \Big[\hat{H}, \rho\Big]. \tag{2.1.4}$$

The first order time differential nature of these dynamics allows one to uniquely determine the future time evolution of any defined state. In this way the evolution of a quantum system is similar to that of classical systems in that they are both perfectly deterministic.

The time evolution of a statevector $|\psi(t_0)\rangle$ at some initial time t_0 to a later time t is then given by the solution to the Schrödinger equation.

$$|\psi(t)\rangle = \hat{U}(t, t_0)\,|\psi(t_0)\rangle\,, \tag{2.1.5}$$

where $\hat{U}(t, t_0)$ is the time evolution operator given by

$$\hat{U}(t, t_0) = e^{-i\hat{H}(t-t_0)/\hbar} \tag{2.1.6}$$

in the case in which the Hamiltonian $\hat{H}$ is time independent.

An important feature of the Schrödinger equation is that it is linear. If both the statevectors $|\psi_1\rangle$ and $|\psi_2\rangle$ are solutions to the Schrödinger equation, then the statevector $|\psi\rangle = \alpha\,|\psi_1\rangle + \beta\,|\psi_2\rangle$, where α and β are complex values, also solves the Schrödinger equation. In the case where both the states $|\psi_1\rangle$ and $|\psi_2\rangle$ are possible states of our system S, then so is the superimposed state $|\psi\rangle$ a possible state of our system. This is the famed, and troublesome, superposition principle.

This allows us to write a state of the system as a linear combination of the eigenstates of a chosen observable $\hat{O}$,

$$|\psi\rangle = \sum_n \alpha_n\,|o_n\rangle\,, \tag{2.1.7}$$

where the coefficients α_n are given by $\alpha_n = \langle o_n|\psi\rangle$ which we can write as a function $\psi(o_n) = \langle o_n|\psi\rangle$ of the eigenvalues of the observable $\hat{O}$. This we refer to as the wavefunction. This terminology follows from the wave-like nature with which the state of our system evolves in time.

It would be noted by most that the final so called collapse postulate and Born's rule have not been included in this list. This is intentional and it shall be introduced below in the context of the measurement process. While their results are not thought to be controversial, interpretations that follow from it are.

So far nothing has been said that is not generally accepted by most physicists. Quantum mechanical systems have consistently been shown to be governed by the rules laid out here. In fact, the field of quantum mechanics has resulted in some of the most accurate predictions of measured quantities to date, which is suggestive of the accuracy of these descriptions. Any attempt to resolve the measurement problem must be able to recreate these dynamics in the appropriate limit.

2.2 Measurement and Wave-Particle Duality

The concept of the superposition principle is fundamentally at odds with what one might expect from the properties of a classical system. It implies that in some way a system can simultaneously have several different values for the same property. Schrödinger's cat can be simultaneously dead *and* alive. The absurdity that Schrödinger attempted to point out with his famous thought experiment is the heart of the measurement problem. Superposition has been successfully demonstrated as a measurable phenomenon, as such, there is no doubt that this "absurdity" is true for microscopic systems described by the laws of quantum mechanics. How then does one reconcile this absurdity with our macroscopic experience?

Another way of showing the apparent absurdity of quantum mechanics is with the double-slit experiment via the contrast of the quantum wave-like nature vs the classical particle-like nature of the system under study. There however the introduction of a measuring apparatus in the form of a which-path measurement plays a vital role in this transition, allowing a discussion of simultaneously dead and alive cats and the role of observation on that state.

In the double-slit experiment, an individual particle is made to propagate through a barrier with only two slits in it. In this way, the path of the particle is limited to those that pass through these slits. Since both paths are equally valid options for the particle, we describe the state of the particle at the barrier in the most general sense, as a superposition of both paths, $|s\rangle = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle)$, where the states $|L\rangle$ and $|R\rangle$ are the states of having passed through a left or right slit respectively.

Normally one would then have the particles strike a detection screen some distance beyond the barrier with slits that would then record a position for the particle. This would be

repeated with separate independent particles to build up a pattern of more and less dense regions of particle positions on the detector screen. However, in order to avoid introducing the detector screen as a measuring apparatus before defining the action of a measuring apparatus on the system, we shall instead make use of a density function $\rho(x)$ as a description of the system along the location of the screen.

We define the density function $\rho(x)$ as a real-valued non-negative function of position x on a plane some distance beyond the barrier where traditionally the detector screen is placed. The density function is then interpreted as a measure of the state of the system at position x. It is given in terms of the wavefunction which describes a state in terms of the eigenvector $|x\rangle$ of the position observable $\hat{x}$ with associated eigenvalue x. We then have that, in the case of a pure state $|\phi\rangle$, the density function is given as $\rho(x) = |\phi(x)|^2 = |\langle x|\phi\rangle|^2$. In the case of a mixed state, we can relate the density function to the density matrix via the definition

$$\rho(x) = \langle x| \rho |x\rangle = \sum_i p_i |\langle x|\phi_i\rangle|^2, \tag{2.2.1}$$

where p_i is the classical probability of a particular state ϕ_i.

The density function would usually be defined in terms of the pattern developed on the detector screen via repeated strikes of independent runs of the experiment, with density referring to the number of strikes per region of space at position x. From this definition, the interpretation of the density function as a probability function describing the probability of the particle's path ending at a particular position seems natural, but the idea of the particle's path ending at a well-defined position is counter to the discussion. Instead, we note the combined nature of classical probability and quantum mechanical complementarity within the description given by the density function, and will return to the probabilistic interpretation suggested here later.

Returning now to our double-slit experiment with initial state given by the superposition of a particle travelling through the left and right slits. If one were to assume that our particles behaved in a classical manner, perhaps with similar dynamics as a thrown ball, then we would expect a mixed density function at the position of the screen with the particle taking each path with equal probability and the function being described by two localized peaks directly

behind each slit. As such the density function $\rho(x)$ would be given by

$$\rho(x) = \frac{1}{2}(|\langle x|L\rangle|^2 + |\langle x|R\rangle|^2). \tag{2.2.2}$$

This, however, is an old experiment and few nowadays would be surprised by the revelation that this is not what we see in practise. Instead what is found is a pattern of alternating bands of high and low density: an interference pattern. This pattern is associated with two waves that propagate outwards from the positions of the slits and interfere,

$$\rho(x) = \frac{1}{2}|\langle x|L\rangle + \langle x|R\rangle|^2, \tag{2.2.3}$$

$$= \frac{1}{2}|\langle x|L\rangle|^2 + \frac{1}{2}|\langle x|R\rangle|^2 + \mathrm{Re}\{\langle x|L\rangle\langle R|x\rangle\}, \tag{2.2.4}$$

hence we find that quantum mechanical systems behave in a wave-like manner in contrast to the expected particle manner expected by the classical description above in equation (2.2.2). The second line in equation (2.2.3) emphasises the interference that is found in this experiment via the third term, $\mathrm{Re}\{\langle x|L\rangle\langle R|x\rangle\}$.

The idea that matter somehow is wavelike in nature was rather profound upon its discovery, as one might expect. If all objects are fundamentally made of particles such as those which we just suggested behaved in a wave-like manner, then this result tells of a world in which the dispersion of massive objects would be a normal occurrence, which is not the case. Perhaps then our particles instead only interfere with the possibility of another path, but are still non-wave-like point particles in nature (as might be argued when one remembers the original experiment in which the particle strikes a screen and only a single point position is registered). Indeed this line of thought might lead to the de Broglie–Bohm Pilot wave interpretation of quantum mechanics. If this is the case, then one might ask which path, in particular, the particle takes by attempting to measure its presence immediately beyond one of the slits in the barrier.

It is at this moment where the micro- and macroscopic meet: the moment of measurement, since any measuring apparatus at some point would be considered macroscopic by anyone's standard, if not the apparatus employed, then the sensory organs of the experimenter themself. To describe this moment we make use of von Neumann's ideal measurement scheme [90]. This was the first explicit description of the measurement process in terms of the rules

of quantum mechanics and is still used to this day with most textbooks on foundational quantum mechanics referring to it.

To this end, we introduce the (macroscopic) system A which is designed as an apparatus to measure the property O of a system with the Hilbert space $\mathcal{H}_S$. Here the measured property is the path of the particle through the slits in the barrier. The Hilbert space $\mathcal{H}_A$ of this apparatus includes some "ready state" $|A_0\rangle$ that the apparatus takes on initially before any interaction with a system, as well as, in this case, two mutually orthogonal states $|A_L\rangle$ and $|A_R\rangle$, each corresponding to a different distinguishable configuration of the apparatus from which an experimenter making use of this apparatus might be able to tell the final value of the measurement. These configurations are usually called "pointer states" as they have been historically associated with different positions of a pointer along a dial upon the apparatus. It is then assumed that the nature of the interaction of the apparatus with the system is linear and that it results in a perfect correlation between the eigenstates $|L\rangle$ and $|R\rangle$ being measured and the final configuration of the apparatus, $|A_L\rangle$ and $|A_R\rangle$. In the case of the particle taking the left path, the combined particle-apparatus system evolves according to

$$|L\rangle\,|A_0\rangle \to |L\rangle\,|A_L\rangle\,, \tag{2.2.5}$$

compared to the case in which the particle takes the right path,

$$|R\rangle\,|A_0\rangle \to |R\rangle\,|A_R\rangle\,. \tag{2.2.6}$$

From this we have an apparatus that can be used to identify the path of the particle. It is important here that this measurement of the path of the particle does not effect the end state of the particle at all, as is the case in an ideal measurement.

If we now include this new apparatus A and let the combined system-apparatus state in the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_A$ evolve, we result in

$$\begin{aligned} |s\rangle\,|A_0\rangle &= \frac{1}{\sqrt{2}}\Big(\,|L\rangle + |R\rangle\,\Big)\,|A_0\rangle\,, \\ &\to \frac{1}{\sqrt{2}}\Big(\,|L\rangle\,|A_L\rangle + |R\rangle\,|A_R\rangle\,\Big). \end{aligned} \tag{2.2.7}$$

The final state given here is often referred to as pre-measurement to emphasise that this cannot lead to a case where measurement has yet been performed. This is clear from the right-hand side of equation (2.2.7) above where the experimenter cannot conclude any property

of the system as the apparatus is found to be in a superposition of distinguishable states. While this in itself might not be an issue in the context of quantum mechanics, it is in rather stark contrast to an experimenter's experience of apparatuses making measurements in their daily lives.

Investigating what this means for our particle beyond this apparatus, we no longer find that the particle behaves as it did before. Instead, in practice, we find that the pattern described by the density function after measuring the path of the particle is instead that of the form of equation (2.2.2) which describes a classical particle. The emergence of this behaviour becomes apparent in the description of the density function,

$$\rho(x) = \frac{1}{2}|\langle x|L\rangle|^2 + \frac{1}{2}|\langle x|R\rangle|^2 + \mathrm{Re}\{\langle x|L\rangle\langle R|x\rangle\langle A_L|A_R\rangle\}, \tag{2.2.8}$$

when one takes into account the orthogonality of the apparatus states, $\langle A_L|A_R\rangle = 0$, in the last term representing the interference. The exact derivation of this result is left for later in section 3.2.

This change in behaviour of the system has colloquially been referred to as particle-wave duality. As long as we do not ask questions relating to a particle description of the system (such as its path), the state of a system behaves in a wave-like manner, but the moment such a question is asked the system behaves like a particle. This is the *complementary* referred to by Niels Bohr between the particle associated which path information and the wave-like interference we see. In this way we can write two limiting cases for the nature of our particle passing through the double-slit experiment:

- The *wave* case in which no measurement is performed and we find a system that can be described with a wave-like evolution, taking both paths simultaneously resulting in interference, as described in equation (2.2.3). This is the case of quantum behaviour with the particle experiencing superposition and evolving according to the Schrödinger equation.

- The *particle* case in which a measurement is made. In this case no interference is seen and the outcome of the experiment is reminiscent of classical particles passing through only one slit to end up in a localized position beyond with equal probability,

as described by a density function of outcomes given by equation (2.2.2). This would be associated with a classical case where interference does not occur and systems are described by well localized states.

We then see that the nature of a quantum mechanical system exists in a duality. The neat deterministic, linear evolution described by the Schrödinger equation associated with the wave-like nature suggested above, compared to the non-deterministic stochastic evolution upon introduction of a measuring apparatus. Even so, the experimenter in the above case is still left with no exact outcome to interpret. While the density function now describes a mixed state with classical probabilities upon the introduction of the which-path detector, the particle-apparatus system itself is still left in a superposition given by equation (2.2.7).

The standard way out of this conundrum is the final postulate eluded to in the previous section: the so-called collapse postulate.

4. Upon measurement of a property with associated observable $\hat{O}$, the state of the measured system S undergoes a drastic and sudden change, sometime referred to as collapse or reduction, by the end of which its statevector becomes a randomly chosen eigenvector of the measured observable,

$$|\phi\rangle \rightarrow |o_n\rangle\,, \tag{2.2.9}$$

The probability with which the system might undergo the reduction to a particular final eigenstate is given by Born's rule,

$$P[o_n] = |\langle o_n|\psi\rangle|^2 = \mathrm{Tr}\Big(\hat{Z}(o_n)\rho\Big), \tag{2.2.10}$$

where $\hat{Z}(o_n) = |o_n\rangle\langle o_n|$ is the projection operator to the eiegnvector $|o_n\rangle$ associated with the eigenvalue o_n. In the context of the density matrix, the reduction is given as

$$\rho \rightarrow \frac{\hat{Z}(o_n)\rho\hat{Z}(o_n)}{\mathrm{Tr}\Big(\hat{Z}(o_n)\rho\Big)}. \tag{2.2.11}$$

Now, with this final postulate added, admittedly in a rather *ad hoc* fashion, the pre-measurement, equation (2.2.7), can be resolved into a case where our experimenter might make sense of the final state of the apparatus. With probability $\frac{1}{2}$, the apparatus is found to be in a final state

$|A_L\rangle$ or $|A_R\rangle$, associated with a system described by the state $|L\rangle$ or $|R\rangle$ respectively. The which-path detector finds the system to have an equal probability of passing through the left or right slit.

One might then ask how to interpret this result in terms of the state of the system before measurement. While it could be said that it is true that the state of the system immediately after measurement does indeed possess a value o_n for the property O, can any similar statement be made in the context of the original system we were interested in? Does asking this question make sense? It is in this way that referring to the process of letting a system interact with some apparatus as "measurement" might be ill-defined if the thing we are measuring (in the classical sense) is, in fact, the system *after* interaction with our apparatus. It is in this same sense that the idea of assigning a "property" to the system makes no sense and cannot be done. How do you assign a system a location when it is equally in many locations? This might very well be the limitations of our classical sensibilities, and by extension language, coming into stark contrast with the alien rules of quantum mechanics. By this framing, the measurement problem is then the insult to those sensibilities.

2.3 The Problem

With the addition of the collapse postulate introduced above, one would be excused for believing that the description of quantum mechanics has no problems. Indeed, the description given by these four standard postulates taught to most physicists is complete in the sense that they have resulted in nearly a century of highly successful descriptions of measurement outcomes from experiments done on quantum systems. In fact, there would seem to be a significant portion of physicists [110, 114] that believe the measurement problem to be a pseudo-problem, with one's personal philosophical perspective being an important contributor of that view. If one is of the standing that quantum mechanics (or modifications to it) should in some way describe reality beyond mere predictions, then the problem of measurement becomes apparent.

It is most common, it would seem, for the problem of measurement to be introduced and defined in the same context with which we introduced the collapse postulate above, i.e. with

regard to the failings of standard quantum mechanics, minus the collapse postulate, to make sense of the process of measurement in a manner that is in accord with our macroscopic classical sensibilities. In fact, the collapse postulate was Bohr, Heisenberg, and von Neumann's patch to fix the measurement problem during the formulation of quantum mechanics. The problem has been shown to include various other issues since then, each still related to the interaction of quantum systems with those systems we would consider classical. Additionally, there are two intimately related aspects to the measurement problem, each a problem in themselves, that can be lost in the more common ways of introducing it. As such, we shall attempt to define the problem of measurement in quantum mechanics with as little ambiguity as possible:

- **Problem of Macro-Objectification:** For a complex system constructed of parts that have been shown to evolve according to the linear Schrödinger equation, why is it that said complex system does not also evolve according to the Schrödinger equation?

- **Problem of Outcomes:** Upon interaction of a quantum system with some apparatus whose purpose it is to measure some observable of the system, why is it that the system-apparatus interaction ends with the system in a unique pure eigenstate of the measured observable?

  - ☐ **Emergence of Born's rule:** Why is the probability of the system evolving to end in a particular state upon interaction with a measuring apparatus given by the Born's rule?

  - ☐ **Basis Ambiguity:** This is discussed in more depth in its own section below, section 2.3.1. In short, it is the question as to why we happen to experience our reality in a particular basis when there is no preference for it given in the descriptions of quantum mechanics. Any other basis is just as valid a description of reality.

  - ☐ **Final State Objectivity:** If we are to assume consecutive measurements are performed on a prepared system with no prior agreement as to which basis should be measured, then the results that would be obtained in this way would be subjec-

tive in that they would be specific to each measurement. This is further discussed and defined in section 2.3.2 below.

The manner in which the individual problems that together make up the measurement problem are listed here is not intended to infer any hierarchical relationship between them. The Problem of Macro-Objectification and Problem of Outcomes are the main aspects of the problem, however, these issues are closely related and it is not uncommon for authors to not differentiate them at all. The issues of the Emergence of Born's rule, Basis Ambiguity, and Final State Objectivity should not be interpreted as subordinate but rather secondary related problems, which have been identified as completely separate issues in some places. They have been included here as they result from the core measurement problem. While the measurement problem is at its heart a single problem, it is one with many aspects that can be interpreted as separate issues in themselves.

2.3.1 Basis Ambiguity

In short, this is the question of why we seem to find things in a particular basis, such as the position basis we are familiar with, rather than any other, given that a quantum mechanical system can be described in any basis. There is no precedent for any particular basis in our current description of quantum mechanics without the collapse postulate.

To demonstrate this we introduce a system S with the Hilbert space $\mathcal{H}_S$ prepared in some state $|s\rangle = \sum_i \alpha_i |a_i\rangle$. Some apparatus A is devised to measure the observable $\hat{A}$ with eigenstates $\{|a_i\rangle\}$ upon interaction with the system. The apparatus is designed to evolve from some ready state $|A_0\rangle$ to one of a set of pointer states $\{|A_i\rangle\}$, each associated with a particular eigenstate of the observable being measured, $|a_i\rangle |A_0\rangle \to |a_i\rangle |A_i\rangle$, in accord with von Neumann's ideal measurement scheme.

Having the prepared system then interact with the apparatus, we get the superposition state

$$|s\rangle |A_0\rangle = \left(\sum_i \alpha_i |a_i\rangle\right) |A_0\rangle \to \sum_i \alpha_i |a_i\rangle |A_i\rangle , \tag{2.3.1}$$

in parallel with what was found in equation (2.2.7). The problem arises when it is noted that this outcome is not unique and that in general any pointer state can be devised for the

same apparatus such that any observable of the system Hilbert space can be measured with this same physical apparatus. Nothing specifies the particular pointer states $\{|A_i\rangle\}$ as in any way special. It follows from treating the pointer of our apparatus as a quantum object that the Hilbert space $\mathcal{H}_A$ of the apparatus pointer states can be described in a basis other than $\{|A_i\rangle\}$. For some observable $\hat{B}$ with eigenstates $\{|b_i\rangle\}$ in $\mathcal{H}_\mathcal{S}$, there could exist associated pointer states $\{|B_i\rangle\}$ of the apparatus such that

$$\sum_i \alpha_i \, |a_i\rangle \, |A_i\rangle = \sum_i \beta_i \, |b_i\rangle \, |B_i\rangle \, , \tag{2.3.2}$$

where the unitary basis transformation has just been absorbed into the apparatus states. The same physical apparatus has formed one-to-one correlations with both the $\{|a_i\rangle\}$ and $\{|b_i\rangle\}$ states, implying that this same apparatus has performed measurements in both bases simultaneously with no change in physical configuration.

In the context of the double split experiment discussed above where the measuring apparatus couples with the system resulting in the state $\frac{1}{\sqrt{2}}\big(\, |L\rangle \, |A_L\rangle + |R\rangle \, |A_R\rangle \, \big)$, we could have instead described the system in the orthonormal basis $\{|\pm\rangle = \frac{1}{\sqrt{2}}\big(\, |L\rangle \pm |R\rangle \, \big)\}$. This would amount to measuring whether the particle passed through any slit at all ($|+\rangle$) or neither slit ($|-\rangle$). The Hilbert space $\mathcal{H}_A$ of the apparatus pointer could then be described in the associated basis $\{|A_\pm\rangle = \frac{1}{\sqrt{2}}\big(\, |A_L\rangle \pm |A_R\rangle \, \big)\}$ completely legitimately according to the quantum nature of these states. The final state of the interaction of the particle with the apparatus can then be shown to be described as

$$\frac{1}{\sqrt{2}}\big(\, |L\rangle \, |A_L\rangle + |R\rangle \, |A_R\rangle \, \big) = \frac{1}{\sqrt{2}}\big(\, |+\rangle \, |A_+\rangle + |-\rangle \, |A_-\rangle \, \big). \tag{2.3.3}$$

If one were to then investigate the interference pattern produced by the system in each of these bases, expressed via the final term in equation (2.2.8), $\mathrm{Re}\{\langle x|L\rangle \, \langle R|x\rangle \, \langle A_L|A_R\rangle\}$ and $\mathrm{Re}\{\langle x'|+\rangle \, \langle -|x'\rangle \, \langle A_+|A_-\rangle\}$ for the basis $\{|L\rangle \, , |R\rangle\}$ and $\{|+\rangle \, , |-\rangle\}$ respectively, one would find that both end up being simultaneously suppressed by the interaction with the apparatus, a rather impossible notion.

In practice this problem is lessened somewhat via the requirement that the apparatus pointer states must be chosen to be orthogonal, $\langle A_i|A_j\rangle = 0$ for $i \neq j$, as to ensure that the pointer states are classically distinguishable. Additionally, an arbitrary choice of apparatus

states $\{|B_i\rangle\}$ may measure an observable $\hat{B}$ that is not necessarily Hermitian. As such one might require that the system states associated with each pointer state be mutually orthogonal as well, $\langle a_i|a_j\rangle = 0$ for $i \neq j$. With these requirements and the normalization restriction on the coefficients, $\sum_i \alpha_i^2 = 1$, it then follows from the biorthogonal decomposition theorem, or so-called Schmidt theorem, that for *real* and *distinct* ($\alpha_i \neq \alpha_j \forall i \neq j$) coefficients, the outcome of the measurement process, equation (2.3.1), is unique. Regardless of these restrictions, there still remains a large set of outcome states $\sum_i \beta_i |b_i\rangle |B_i\rangle$ that are not unique, as shown by the simple example of the bipartite double split experiment with equal coefficients mentioned here.

This equivalence of possible bases in which to express the outcome state means that once the measurement process has been performed in the basis of one observable, it would seem that it is simultaneously performed in the basis of a large set of hypothetical other observables, all without any physical change to the measuring apparatus, and regardless of the commutation relations between these observables. This is fundamentally at odds with quantum mechanics in which we cannot simultaneously measure two non-commuting observables. This is stated more physically as the contradictory attempt to suppress interference in more than one basis simultaneously. This is the problem of Basis Ambiguity. There is no way of choosing a preferred basis in which to perform a given measurement that does not lead to simultaneous bases being measured in.

2.3.2 Final State Objectivity

It has been shown that the process of measurement acts to disturb the system of interest regardless of the inclusion of the collapse postulate. The act destroys any interference between superimposed states, leaving the apparatus in a mixture of pointer states. The problem of objectivity arises if one then includes the action of a series of ignorant observers who wish to consecutively probe the state of the apparatus as a means of determining the state of the system of interest by performing their own measurement on the apparatus, adding themselves to the von Neumann measurement chain. Here ignorant refers to them having no knowledge of the choice of pointer states.

The first of such a measurement performed by an observer on the apparatus would gen-

erally be done in an arbitrary basis due to the observer's ignorance of the pointer states. As a result of this measurement, the observer will receive a result in the measured basis and the state of the apparatus would then be changed to a mixture in this new basis which would be different to that mixture with which we started. Any measurement performed on the system now would then obtain some different result that is incompatible with the result obtained before, and once again change the system. This subjective measurement experience is fundamentally at odds with what one might expect from a similar measurement process performed in the classical context. There we expect the experience to be objective, for consecutive measurements to obtain consistent results.

This could be avoided if we assume the observers to not initially be ignorant and all measurements performed in the same pointer basis, in which case each observer would receive consistent results between them with each measurement confirming the result of the previous. While this creates a scenario that could be considered objective, it does require that observers have some knowledge about the pointer states and a consensus amongst them to only ever measure in said basis. While this scheme's apparent objectivity might seem attractive, the requirement for all observers to have prior knowledge to instruct their measurements is not ideal.

These two scenarios demonstrate the lack of final state objectivity within the current description of quantum mechanics. The process of measurement within this description results in an effectively subjective experience for ignorant observers, each finding the final system-apparatus in a different outcome state. This is far from what one might have in mind for a reliable apparatus that results in an objective *classical* outcome.

This thought is mirrored in Einstein, Podolsky, and Rosen (EPR) [59] in their famous paper on entanglement where they refer to their element of physical reality,

> If, without in any way disturbing a system, we can predict with certainty (i.e. with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.

This leads us to define an operational definition of objectivity, which we shall borrow from Ollivier, Poulin, and Zurek [92] who based their definition on that element of physical reality

defined by EPR in a rather satisfactory way and is consistent with the objectivity we find lacking in our discussion so far. There they define an objective property of the system as

1. simultaneously accessible to many observers,
2. who are able to find out what it is without prior knowledge about the system of interest, and
3. who can arrive at a consensus about it without prior agreement.

Applying this definition to the process of measurement, one can demonstrate this definition's relation to the element of physical reality mentioned by EPR.

2.4 The Role of Linearity and Unitarity

We have discussed the problem of measurement and attempted to define it as clearly as possible, going to some length to demonstrate the intrinsic duality in our current ways of describing quantum mechanics. This we illustrated in the context of the double-slit experiment and the apparent change in the nature of the quantum system upon introduction of a measuring apparatus. However, this duality is best encapsulated in the contradictory stochastic nonlinear evolution described during the process of measurement as described by the collapse postulate. The dynamics of the quantum system during the process of measurement is completely at odds with the linear deterministic nature of evolution described at any other time by the Schrödinger equation.

Careful scrutiny of the measurement problem reveals that there are two related features of quantum mechanics that underpin the measurement problem and with which it is not possible to avoid the difficulties it poses. These two features are linearity and unitarity of time evolution. With this, we mean that the time evolution operator is both linear and unitary.

- Linearity allows for system states to exist in a superposition, i.e. any linear superposition of solutions to the Schrödinger equation is again a solution. When coupled to a macroscopic apparatus which ought to behave classically and whose state is dependent on that of the system, the end result of the evolution is a superposition of a macroscopic

system states as described by equation (2.2.7), a property fundamentally at odds with classical descriptions. This is essentially the problem of Macro-Objectification.

- Unitarity of the systems dynamics prevents a state initially in a superposition of eigenstates of some observable from evolving into a single eigenstate of the observable. The need to preserve the inner product of the state with that of its basis vectors prevents a final state described by equation (2.2.9). This impossibility of evolving to a single basis state is the problem of Outcomes.

To avoid the measurement problem, we must break from these two properties, whether its linearity as argued by [21], Unitarity, or both. This, of course, represents a fundamental modification of quantum mechanics.

Of course, these notions are not unrelated, one cannot break linearity and unitarity independently as unitarity implies linearity and, subsequently, non-linearity implies non-unitarity. There are therefore only two scenarios possible: The dynamics might be linear but non-unitary, or non-linear and non-unitary. Both these options include a break from unitarity, reducing this to a choice of whether to break linearity or not. Regardless, a break from unitarity is not taken lightly as it would mean no guarantee that conserved properties of the system remain so. The choice to break linearity is also problematic as any proposition that does so must recreate superposition effects for systems thought of as quantum in nature. Indeed, any proposition must be able to recreate the traditional dynamics given by the Schrödinger equation in the appropriate limit, which is still the most accurate theory to date.

In this investigation we have chosen two fields of study that are intimately related to the transition from a quantum description to one that is classical in nature. Each represents one of the possible ways of breaking from linearity and unitarity listed above. Decoherence, while not a solution to the measurement problem as it fails to identify a unique outcome, is an example of dynamics where unitarity has been broken via the inclusion of an environmental bath while the dynamics are still linear. Alternatively, Dynamical Reduction programs offer an example of dynamics where linearity has been intentionally broken in an attempt to resolve the measurement problem. In so doing, the program becomes neither linear nor unitary. Both

these programmes have been studied for their ability to resolve aspects of the quantum to classical transition and as such are investigated further in their own sections below. While each has there shortfalls, they are also the most popular attempts beside the still dominant Copenhagen interpretation.

3. Environmental Decoherence

As might be interpreted by the phenomenon's name, environmental decoherence is the effective decay of coherence within a system of interest due to the interaction and resulting coupling to environmental degrees of freedom [109]. This recognition that realistic quantum systems can never be truly isolated from their environment was first pointed out by Zeh in 1970 [126], and has since been massively expanded by Zurek and others [108, 107, 132, 129, 127]. It was recognised that, upon introduction of an environment, the coherence between system states could not be observed with any sort of local measurement on the system, being lost in the environment. Even in the ideal case where the interaction has no effect on the dynamics of the system, it is the coupling that causes this suppression of quantum coherence in the system.

This is a basic summary of the decoherence scheme. It describes how the assumption of an open system then affects what we are actually capable of measuring with any local measurement. If a measurement of the system along with all environmental degrees of freedom that had in any way interacted with the system, potentially the entire universe, could feasibly be made one would be able to reconstruct the coherence that was initially lost, but such a measurement is impossible to achieve. The coherence that was suppressed by the interaction with the environment is effectively lost, making the process essentially irreversible.

Decoherence applies the von Neumann measurement scheme but introduces the environment as a third system in a von Neumann chain that measures the apparatus. In this way, the environmental decoherence scheme stays within the established structure of standard quantum mechanics, allowing for a simultaneously unitary and linear approach to the quantum-to-classical transition. Of course, these properties only apply at the level of the combined system-environment state vector as any description of an open system by itself would not be unitary in general. Being limited to only the perspective of the system, we obtain an effectively non-unitary evolution.

A general description of the formalism of the environmental decoherence scheme will be given below in section 3.2 after it is illustrated in section 3.1 how that environment might

cause decoherence via many successive weak measurements on the system. The decoherence scheme in no way solves the measurement problem in quantum mechanics completely [48, 68, 5] as it does not resolve an outcome during measurement, merely a "menu" from which an outcome might be selected [60]. However, it is intimately related to the quantum-to-classical transition. This apparent suppression of interference caused by the interaction of our system of interest with the environment is decoherence's primary result and a possible solution to the inability to observe superpositions of macroscopic systems, resolving the problem of Macro-Objectification in the Measurement Problem [108]. The consequence of this result is the selection of a set of *pointer states* that survive the decoherence process and hence are a natural basis in which to describe any state, offering a solution to the problem of basis ambiguity [132, 109, 130]. This behaviour of decoherence to dynamically select these preferred pointer states has been referred to in the literature as environmentally induced superselection, or *einselection*, and is further detailed in section 3.3 below.

Extension of the role of the environment as a means for the propagation of information about the state of our system, referred to as "the environment as a witness", allows for multiple observers to obtain information about the state of a system within this pointer basis in such a way that the observers might all agree on the result. The process by which observers gain said information has been coined as *quantum Darwinism* due to the selection of the most robust and redundant, i.e. "strongest", states to be stored in the environment to be retrieved by an observer [133]. Many aspects of quantum Darwinism are still being studied and the theory is in its infancy, but the premise is described in section 3.4.

3.1 Weak Measurements and the Environment

The classical idea of an apparatus is that of one with macroscopically distinguishable output states that perfectly couple to the eigenstates of a measured observable. It is possible, however, to instead have an apparatus that only partially probes the state of a system. Such measurement techniques are referred to as weak measurements as the apparatus only weakly interacts with the system. The output states of an apparatus designed to perform such a measurement no longer offer perfect information about the state of the system as such the

apparatus states are taken to be no longer orthogonal, $\langle A_i|A_j\rangle \neq 0$ for some apparatus states where $i \neq j$.

The overlap between the output states of the apparatus then dictates how much information can be gained about the state of the system at the cost of the quantum behaviour of the system. The interference term in equation (2.2.8), from our discussion on the double split experiment in section 2.2, $\mathrm{Re}\{\langle x|L\rangle \langle R|x\rangle \langle A_L|A_R\rangle\}$, demonstrates this. By varying the overlap $\langle A_L|A_R\rangle$ of the apparatus output states, a continuous transition from a quantum description to one of more classical dynamics can be achieved. In the region between the two extremes, it is still possible to obtain partial information about the system state while maintaining a portion of its quantum behaviour. The interference term, representing the quantum coherence in the system, is only suppressed in the case where we extract complete knowledge of the state of the system. This trade-off between interference and information was analyzed by Zurek and Wootters [123] where they showed that one can gain about 90% certainty about the state of the system without losing more than 50% contrast in the interference pattern.

The amount of quantum behaviour we see in a system is directly related to the amount of information about that system that can be stored in some second system entangled with the first [61, 123], regardless of whether that information is obtainable [109]. If the role of this second, weakly interacting system could be fulfilled by subsystems of the environment, such as individual photons being scattered with a scattering cross-section weakly dependent on the state of the system, then one could be convinced that the collective effect of the environment as a whole might begin to induce decay of the coherence in the system.

To illustrate this idea, let us start at some initial time $t = t_0$ with some system of interest $S = |s(t_0)\rangle = \alpha\,|1\rangle + \beta\,|2\rangle$. It might or might not be macroscopic, it does not matter for this description, but we shall assume it is subject to the laws of quantum mechanics regardless. Our system then interacts with a single environmental degree of freedom $E = |e(t_0)\rangle = |e_0\rangle$. This might be a single air molecule or photon scattering off our system or a single electron spin whose magnetic field our system interacts with. Regardless, the interaction happens with our $\{|1\rangle, |2\rangle\}$ states in which we described our system above. Our combined system-environment state then evolves according to the following dynamics;

$$|i\rangle\,|e_0\rangle \longrightarrow |i\rangle\,|e_i\rangle\,. \tag{3.1.1}$$

Here $i = 1, 2$ and $|e_i\rangle$ is the final states of the environmental degree of freedom after the interaction at some time $t = t_1$.

For this interaction we assume a weak coupling that gets stronger in proportion to the duration of the interaction, $\Delta t = t_1 - t_0$. As such we have

$$\langle e_1|e_2\rangle = 1 - \epsilon\Delta t, \tag{3.1.2}$$

where the coupling constant ϵ is chosen such that $0 < \epsilon\Delta t \ll 1$ is very small. In general this form will depend on the nature of the interaction in question, but this is a rather useful form for this demonstration.

For the given initial system of interest, the final system-environment state then is given by

$$|s(t_0)\rangle\,|e(t_0)\rangle = (\alpha\,|1\rangle + \beta\,|2\rangle)\,|e_0\rangle \tag{3.1.3}$$

$$\rightarrow \alpha\,|1\rangle\,|e_1\rangle + \beta\,|2\rangle\,|e_2\rangle \tag{3.1.4}$$

$$= |se(t_1)\rangle\,, \tag{3.1.5}$$

where $|se(t_1)\rangle$ is the final inseparable system-environment state. In order to describe the system alone, we are required to transition to a density matrix description, $\hat{\rho}_S(t)$. This is due to there being unobtainable information about the system in the environment resulting in a description of the system alone being mixed, which is beyond the ability of a state vector to describe. As such we have that the state of the system is given via the partial trace of the density matrix describing the total system-environment state $\hat{\rho}_{SE}(t) = |se(t)\rangle\,\langle se(t)|$. Performing this partial trace gives

$$\hat{\rho}_S(t_1) = \mathrm{Tr}_E(|se(t_1)\rangle\,\langle se(t_1)|) = \begin{bmatrix} |\alpha|^2 & \alpha\beta^*(1-\epsilon\Delta t) \\ \alpha^*\beta(1-\epsilon\Delta t) & |\beta|^2 \end{bmatrix}. \tag{3.1.6}$$

Here we see the off-diagonal terms of the system density matrix are modified by the overlap of the environment states. Assuming that the interaction is week $0 < \epsilon\Delta t \ll 1$, one can see that the effect of a single interaction might only suppress the interference terms (the off-diagonals in the density matrix) by a negligible amount.

We now assume that our system interacts sequentially with additional environmental degrees of freedom, for a total of N interactions. Assuming each of these additional interactions

is identical to the first, resulting in the full interaction with all N environmental degrees of freedom ending at some time t_N, we can follow nearly the same procedure as above for each interaction. The only difference is the initial system for the i-th interaction would be given by the state described by the final reduced density matrix of the $(i-1)$-th interaction, $\hat{\rho}_S(t_{i-1})$. As such, after all N interactions, the density matrix of the system will be given by

$$\hat{\rho}_S(t_N) = \begin{bmatrix} |\alpha|^2 & \alpha\beta^*(1-\epsilon\Delta t)^N \\ \alpha^*\beta(1-\epsilon\Delta t)^N & |\beta|^2 \end{bmatrix}. \tag{3.1.7}$$

This result allows us to see overlap of the environmental states for the combined environment of all N subsystems,

$$\langle E_i|E_j\rangle = (1-\epsilon\Delta t)^N. \tag{3.1.8}$$

Here the state of the full environment, including all environmental degrees of freedom that interacted with the system, is then $|E_j\rangle = \bigotimes_{i=0}^{N} \left|e_j^{(i)}\right\rangle$, with $\left|e_j^{(i)}\right\rangle$ being the i-th environmental degree of freedom to interact with the system. If we take each interaction as consecutive such that the total duration of the complete system-environment interaction is $t = t_N - t_1$ with $t = N\Delta t$, we can make the substitution $\Delta t = \frac{t}{N}$. If our system happened to be a mote of dust suspended in the air as molecules scatter off it, the number of interactions would be of the order 10^{19}. As such, taking the limit of $N \to \infty$ seems appropriate. Under this limit, we see that the above overlap takes the form

$$\langle E_i|E_j\rangle = e^{-\frac{t}{\tau_{i,j}}}. \tag{3.1.9}$$

We now see that the interference terms decay exponentially for large numbers of very weak interactions with a characteristic time constant $\tau_{i,j}$ that might depend on the nature of the states i and j [109].

This decay of the inner-product of the states of the complete environment results in these states becoming macroscopically distinguishable. Additionally, this process only happens in this particular basis of environmental states, which depends on the nature of the interaction with the system. This gives the particular observable eigenstates that result in this decay a status as being preferred, as will be discussed in section 3.3.

While this derivation is far from rigorous, it offers an illustration of how the environment, through constant weak interactions with the system, can cause the decoherence of the systems

states. The final result is similar to what one would expect from some existing models [109, 107, 85, 95, 89, 83]. One such model is that of decoherence of the position of some system due to scattering of environmental particles. Using this model, one finds that for a dust mote of the scale 10^{-5}m, decoherence occurs within a timescale of 10^{-23} seconds due to scattering of air molecules at standard pressure [107], demonstrating how quickly the process of decoherence occurs.

Here we have shown that regardless of how weak individual interactions between the degrees freedom of the environment and our system might be, after some (usually very short) time our system will effectively decohere as the system's coherence is carried away by the environment and cannot be locally measured.

3.2 The Basic Decoherence Scheme

Having illustrated the effectiveness of the environment as a measuring apparatus, which has effectively orthogonal states shortly after interaction with the system, we shall give the formalism of the Environmental Decoherence scheme. This follows similarly the measurement by the environment described above [107, 109, 131, 130].

We start with a system of interest that has been prepared in some state to be measured, $|s\rangle = \sum_i s_i |o_i\rangle$. Here the system is described in the basis of some observable $\hat{O}|o_i\rangle = o_i |o_i\rangle$. The environment, described by some initial state $|E_0\rangle$, is then introduced. The environment is then allowed to interact with our system of interest. This interaction will have some observable on which it depends which can be found from the interaction Hamiltonian. For our sake, we shall assume this observable is the already defined observable $\hat{O}$. The combined system-environment state evolves over an appropriate amount of time according to

$$|o_i\rangle |E_0\rangle \to |o_i\rangle |E_i\rangle , \tag{3.2.1}$$

where $|E_i\rangle$ is the final state of the environment after interacting with a system in the state $|o_i\rangle$. From the above discussion we can assume that after the interaction is complete we have that the final environmental states are effectively orthogonal, $\langle E_i|E_j\rangle = 0$ for $i \neq j$. It is also taken that the states of the environment $\{|E_i\rangle\}$ form a complete description of the environment.

As a result of the above dynamics, our system of interest and the environment become entangled. This combined state then evolves into

$$|s\rangle |E_0\rangle = \left(\sum_i s_i |o_i\rangle\right) |E_0\rangle \longrightarrow \sum_i s_i |o_i\rangle |E_i\rangle . \tag{3.2.2}$$

Here we note the inseparability of the system and environment as they become entangled on completion of their interaction. However, the nature of the environment as an information bath, with its exact state being beyond our ability to measure, results in the state of the system alone after interaction with the environment being mixed. Since it is impossible to describe a mixed state with a state vector, we transition to a description using density matrices.

The state of the system and environment are then described by $\hat{\rho}_S(t)$ and $\hat{\rho}_E(t)$ respectively. The combined system-environment state is then described by the density matrix $\hat{\rho}_{SE}(t)$ which initially is the tensor product of the system and environment density matrix $\hat{\rho}_{SE}(t_0) = \hat{\rho}_S(t_0) \otimes \hat{\rho}_E(t_0)$, but as $\hat{\rho}_{SE}(t)$ evolves, the two systems will no longer be separable as they entangle during the course of the interaction. The full evolution given in density matrices is then

$$\begin{aligned} \hat{\rho}_S(t_0) \otimes \hat{\rho}_E(t_0) &= |s\rangle \langle s| \otimes |E_0\rangle \langle E_0| \\ \longrightarrow \hat{\rho}_{SE}(t_f) &= \sum_{i,j} s_i^* s_j |o_i\rangle |E_i\rangle \langle E_j| \langle o_j| , \end{aligned}$$

where t_0 and t_f are the starting and completion times of the interaction. We also note that while $\hat{\rho}_S(t_0)$ is a pure density matrix, we cannot say much about the purity of $\hat{\rho}_E(t_0)$, and then by extension $\hat{\rho}_{SE}(t_0)$, and as such the environment is taken to be mixed.

We are then left with the inseparable system-environment density matrix. To obtain a description of the possible results that might be obtained via a local measurement on the system alone, we perform a partial trace on the combined density matrix, averaging over the

environmental degrees of freedom. Tracing out the states of the environment, we get

$$\begin{aligned}\hat{\rho}_S(t_f) &= \mathrm{Tr}_E(\hat{\rho}_{SE}(t_f)) \\ &= \sum_k \langle E_k| \left(\sum_{i,j} s_i^* s_j \, |o_i\rangle \, |E_i\rangle \, \langle E_j| \, \langle o_j| \right) |E_k\rangle \\ &= \sum_{i,j} s_i^* s_j \, |o_i\rangle \, \langle o_j| \, \langle E_j|E_i\rangle \qquad (3.2.3) \\ &= \sum_i |s_i|^2 \, |o_i\rangle \, \langle o_i| + \sum_{i \neq j} s_i^* s_j \, |o_i\rangle \, \langle o_j| \, \langle E_j|E_i\rangle \, .\end{aligned}$$

In the second line we use the completeness of the environmental states $|E_i\rangle$ and in the last line we separated the diagonal and off diagonal terms to better illustrate the crux of the decoherence scheme. Since it is assumed that $\langle E_i|E_j\rangle \to 0$ as $t \to t_f$, the off-diagonal terms, i.e. the interference terms of the density matrix, get suppressed and are beyond measurement by any local measurement procedure on the system alone. We are then left with the possible outcomes of a measurement on the system being described by the density matrix in the form

$$\hat{\rho}_S(t_f) = \sum_i |s_i|^2 \, |o_i\rangle \, \langle o_i| \, , \qquad (3.2.4)$$

with the Born probabilities of measuring the associated observable eigenvalues along the diagonal, and all off-diagonal terms suppressed to negligible.

Performing a measurement of a hypothetical observable with eigenstates $\{|x_k\rangle\}$ on the system as described by equation (3.2.3), we can then determine the associated probabilities $\rho(x_k)$ of the possible outcomes,

$$\begin{aligned}\rho(x_k) &= \langle x_k| \, \hat{\rho}_S(t_f) \, |x_k\rangle \\ &= \langle x_k| \left(\sum_{i,j} s_i^* s_j \, |o_i\rangle \, \langle o_j| \, \langle E_j|E_i\rangle \right) |x_k\rangle \\ &= \sum_i |s_i|^2 | \, \langle x_k|o_i\rangle \, |^2 + \sum_{i \neq j} \mathrm{Re}\{ s_i^* s_j \, \langle x_k|o_j\rangle \, \langle o_i|x_k\rangle \, \langle E_i|E_j\rangle \}, \\ &= \sum_i |s_i|^2 | \, \langle x_k|o_i\rangle \, |^2 .\end{aligned}$$

Here in the third line I have not yet taken $\langle E_i|E_j\rangle \to 0$ as to reconnect to the result of equation (2.2.3) quoted in section 2.2 where we discussed the double-split experiment. There

this result might be interpreted as each state $\{|o_i\rangle\}$ being positions of slits in a barrier and $|x_k\rangle$ as the discrete position of sensors in an array a distance beyond the barrier. The first term of the third line would then be the probability distribution of a "classical" particle passing through multiple slits in the screen while the second term is the "quantum" interference term that causes the particle to behave as a wave. This second term that describes the quantum contribution disappears as soon as we introduce the effects of the environment $\langle E_i|E_j\rangle \to 0$, showing how the continuous interaction with the environment can drive an object from a quantum "wave" description towards the classical "particle" description. This is environmental decoherence in action [109, 107].

It might be noted that the reduced density matrix takes a form that would satisfy the conditions to be considered a mixed state ($\rho^2 \neq \rho$) such as the statistical mixture represented in equation (2.1.1). This formal similarity does in no way imply that the reduced density matrix used in the decoherence scheme is a *statistical* mixture. To differentiate the nature of the reduced density matrix, it is referred to as an *improper* mixture as compared to the *proper* mixture that can be described as a statistical mixture represented by equation (2.1.1). An improper mixed state describes the case where the system is in fact in a quantum superposition of the component states but a lack of information prevents this from being observable. By contrast, a proper mixed state is a classical statistical mixture where the system is in only one of the component states, but lack of information prevents us from resolving which [67, 109, 127, 44]. This is an important interpretational difference and the reason why the presence of decoherence is not enough to resolve the measurement problem [68, 60]. Decoherence alone still requires some mechanism of collapse to "choose" one of the outcomes [9, 109]. This similarity between improper and proper mixed density matrices is merely a sign that a measurement on the system alone cannot distinguish between the two cases. A density matrix is merely a computational tool for finding the probability of a particular result upon measurement [109].

3.3 Environmentally Induced Superselection

The interaction between the system and environment causes the emergence of a set of environmental states that become macroscopically distinguishable as their inner product is reduced to negligible. The particular basis in which this occurs depends on the nature of the interaction between the system and environment, specifically those environmental states that couple to the eigenstates of the observable being measured. This emergence of a particular set of distinguishable states drives a process dubbed Environmentally Induced Superselection, or Einselection for short, in which a particular unique set of preferred system basis states can be identified in which interference is suppressed [132, 131, 109, 108, 107, 130, 128, 129]. The ability to resolve these states allows the program of environmental decoherence to offer a solution to problems of basis ambiguity which we described in section 2.3.1.

We demonstrate this via the introduction of the environment into the description of basis ambiguity as given in section 2.3.1. Given a system S in the Hilbert space $\mathcal{H}_{\mathcal{S}}$, with an apparatus with ready state $|A_0\rangle$ and pointer states $|A_i\rangle$ in Hilbert space $\mathcal{H}_A$. The apparatus, designed to measure an observable $\hat{A}$ on $\mathcal{H}_S$ with eigenstates $\{|a_i\rangle\}$, then interacts with the system. The state of the system is then expressed as $|s\rangle = \sum_i \alpha_i |a_i\rangle$. The apparatus is assumed to be exposed to an environment in a von Neumann chain,

$$|s\rangle |A_0\rangle \left|E_0^{(A)}\right\rangle = \left(\sum_i \alpha_i |a_i\rangle\right) |A_0\rangle \left|E_0^{(A)}\right\rangle \rightarrow \sum_i \alpha_i |a_i\rangle |A_i\rangle \left|E_i^{(A)}\right\rangle . \tag{3.3.1}$$

We can then make use of the basis ambiguity in the form of equation (2.3.2) in order to rewrite this in the equivalent form

$$|s\rangle |B_0\rangle \left|E_0^{(B)}\right\rangle = \left(\sum_i \beta_i |b_i\rangle\right) |B_0\rangle \left|E_0^{(B)}\right\rangle \rightarrow \sum_i \beta_i |b_i\rangle |B_i\rangle \left|E_i^{(B)}\right\rangle , \tag{3.3.2}$$

where $\left|E_i^{(A)}\right\rangle$ and $\left|E_i^{(B)}\right\rangle$ are the final environmental states upon interaction with the apparatus state $|A_i\rangle$ and $|B_i\rangle$ respectively. Here $\{|B_i\rangle\}$ are some other pointer states of the same apparatus associated with a new observable $\hat{B}$ with eigenstates $\{|b_i\rangle\}$ of the system Hilbert space $\mathcal{H}_{\mathcal{S}}$.

We then assume that the nature of the interaction with the environment drives the overlap of the $\left|E_i^{(A)}\right\rangle$ states to negligible in the process of decoherence, and as such we take

$\left\langle E_i^{(A)} \middle| E_j^{(A)} \right\rangle = \delta_{ij}$. We do not make any such assumptions about the inner-product of the $\left| E_i^{(B)} \right\rangle$ states as the result of the decoherence depends on the nature of the interaction between the apparatus states and the environment. Instead, we define them in terms of the $\left| E_i^{(A)} \right\rangle$ states,

$$\left| E_i^{(B)} \right\rangle = \sum_j e_{ij} \left| E_j^{(A)} \right\rangle . \tag{3.3.3}$$

From this we have that $\left\langle E_i^{(B)} \middle| E_k^{(B)} \right\rangle = \sum_j e_{ij}^* e_{kj}$.

The intention here is to show that the basis in which the suppression of interference performed by the environment is unique, i.e. that for any other observable $\hat{B} \neq \hat{A}$ the suppression of interference is impossible, $\left\langle E_i^{(B)} \middle| E_k^{(B)} \right\rangle = 1$. To do so we solve for the values of e_{ij} and show this to be the case. If instead it were that $\left| \left\langle E_i^{(B)} \middle| E_k^{(B)} \right\rangle \right| < 1$, then, contrary to our statements above, environmental decoherence might not resolve issues of basis ambiguity.

To this end, we study the evolution of the apparatus in the basis of $\{|B_i\rangle\}$. First, however, in order to save the sanity of the writer, we shall drop any mention of the system in the following description as the state of the system is correlated to that of the apparatus. The evolution of the apparatus then reads as follows,

$$\begin{aligned} |A_i\rangle |E_0\rangle &= \left(\sum_j \langle B_j | A_i \rangle |B_j\rangle \right) |E_0\rangle \\ &\to \sum_j \langle B_j | A_i \rangle |B_j\rangle \left| E_j^{(B)} \right\rangle \\ &= \sum_{jk} e_{jk} \langle B_j | A_i \rangle |B_j\rangle \left| E_k^{(A)} \right\rangle \end{aligned} \tag{3.3.4}$$

where we have made use of equation (3.3.3) between the second and third lines. Comparing this to the original evolution of the interaction between apparatus A and the environment in equation (3.3.1), simplified as

$$\begin{aligned} |A_i\rangle |E_0\rangle &\to |A_i\rangle \left| E_i^{(A)} \right\rangle \\ &= \left(\sum_j \langle B_j | A_i \rangle |B_j\rangle \right) \left| E_i^{(A)} \right\rangle , \end{aligned} \tag{3.3.5}$$

and comparing that to the final line of equation (3.3.4) above, which describes the same evolution, we find that $e_{jk} = \delta_{ik} \forall j$ resulting in $\left\langle E_i^{(B)} \middle| E_k^{(B)} \right\rangle = \sum_j e_{ij}^* e_{kj} = \sum_j \delta_{nj} \delta_{nj} = 1$.

This suggests that the set of environmental states that tend to orthogonality are unique given a particular interaction.

To identify these particular preferred states of the system, the predictability sieve approach is used [129, 109, 128]. This is a procedure that sifts through the Hilbert space of the system identifying those states that are most stable upon interaction with the environment. This stability is identified with the ability for these states to behave predictably even after decoherence has occurred. This procedure amounts to seeking those states in the Hilbert space that minimize the von Neumann entropy $\mathrm{Tr}\{\rho \ln \rho\}$, or alternatively maximize the purity $\mathrm{Tr}\{\rho^2\}$, of the system density matrix ρ after interaction with the environment.

The introduction of the environment into the measurement scheme identifies a unique set of preferred system states in which interference is suppressed and as a result can be interpreted by a classical experimenter, offering a solution to the basis ambiguity.

3.4 Quantum Darwinism and the environment as a witness

Decoherence, from the perspective of any measurement made on the system, has resulted in a system that is apparently a mixed state of the pointer basis. While it is not the traditional statistical mixture, the interference suppressed by the environment can be, in theory, recovered by measuring all of the environment, this is in practise an impossible exercise, leaving our system in an improper mixed state. The state of our system, however, is importantly still quantum. A measurement at this point on the system does not resolve unknown information as it would for a proper mixed state, but causes the system to change to the resultant measured state in some process of collapse [9, 109]. Such measurements might not result in objective results unless the observers have some prior knowledge on the state of the system [92, 31, 135]. This is the problem of objectivity mentioned previously in section 2.3.2. If objectivity is a goal for the process of measurement, then we will require a way to gain information about the system without disturbing it such that it is consistent with consequent measurements regardless of initial information of the observer.

The environment as a witness formulation complements the ideas of decoherence by iden-

tifying the information originally treated as lost to the environment to be instead a means to communicate information about the state of the system without disturbing the system. An observer does not usually interact with the system directly, but rather gains information about the system by intercepting information from the portion of the environment they inhabit [92, 93, 31, 30, 109, 130, 135], for example from the photons scatted from an object in their direction. In this case, the information we gain about the system then comes from the environment's measurements of the system instead of our own, then naturally, the information that an observer might obtain will be dictated by what information is contained in the portion of the environment they inhabit. The information that an observer most readily comes by is then that information that is most redundantly stored in the environment. One then is restricted in what conclusions they might make by what information they can acquire. Information that is stored redundantly is then available in several different portions of the environment and to the observers that then inhabit them, giving several different independent observers access to the same information resulting in consistent conclusions between them. This naturally relates the redundancy with which information is stored in the environment with said information being objective according to the criteria for objectivity introduced in section 2.3.2.

The constant measuring of the system by the environment is the environment as a witness scheme. The amplification and propagation of information is what is referred to as quantum Darwinism due to how certain information, namely that in the preferred basis, gets selected for and propagates through the environment in a manner reminiscent of survival of the fittest. This propagation causes this information to be redundantly stored in the environment, allowing separate observers to obtain it. This program has seen some promising initial experimental assessments [39, 34, 36, 120, 38, 104], but the picture of Quantum Darwinism is still not without some holes.

3.4.1 The Basic Concept

To demonstrate the basic ideas of the environment as a witness scheme [92, 31, 134] we consider a system S with Hilbert space $\mathcal{H}_S$. We define some observable with eigenvalue equation $\hat{S}\left|s_i\right\rangle = \sigma_i \left|s_i\right\rangle$ in the system Hilbert space. We then write the system state in the

basis of this observable $|S\rangle = \sum_i s_i |s_i\rangle$. The system then interacts with its environment E with Hilbert space $\mathcal{H}_E$ which is in some initial state $|E_0\rangle$. Upon interaction, the system and environment become entangled in the usual way, $|S\rangle |E_0\rangle \longrightarrow \sum_i s_i |s_i\rangle |E_i\rangle$. The environment is then made up of N disjoint subsystems ϵ_n which are then partitioned into $R \leq N$ environmental fragments with each fragment $\mathcal{F}_r = \bigotimes_{\{n\}_r} \epsilon_n$, $r \in [1, R]$, is then a disjoint subset of the environment, $\mathcal{F}_r \subset \mathcal{H}_E$. Acting on each of these fragments we have some operator $\hat{O}_r$ that couples to the system operator $\hat{S}$ according to some interaction Hamiltonian $H_I(\hat{S}, \hat{\vec{O}})$. The eigenvalues of the operators defined on each fragment, $\hat{O}_r |o_k\rangle_r = o_{k_r} |o_k\rangle_r$, where $k_r \in [1, D_{F_j}]$ label the fragment state (with D_{F_r} the dimension of the fragments Hilbert space $\mathcal{H}_{\mathcal{F}_r}$) and eigenvalue respectively, can then be used to uniquely label the states of the environment $|E\rangle = |o_{k_1}, o_{k_2}, \cdots, o_{k_r}, \cdots, o_{k_R}\rangle = |\vec{o}\rangle$. The pointer states $|E_i\rangle$ are then written as $|E_i\rangle \equiv |o_{i_1}, o_{i_2}, \cdots, o_{i_r}, \cdots, o_{i_R}\rangle = |\vec{o}_i\rangle$ for simplicity.

The distinction between the fragment and the subsystems of the environment is an important one. Subsystems might be individual photons in a photon bath while a fragment might include thousands of photons. The way in which these fragments are constructed are not, however, arbitrary. It is rather core to the environment as a witness scheme that the environment can be written as a separable tensor product of these fragments, $\mathcal{E} = \bigotimes_r^R \mathcal{F}_r$ [92, 31]. In this way, we have that the environment is partitioned into disjoint separable partitions of environmental subsystems. This ensures that observables defined on each fragment commute, $[\hat{O}_r, \hat{O}_{r'}] = \delta_{rr'}$, and as such can be simultaneously measured by individual observers without the outcomes affecting each other.

The system is then introduced to the environment, which then measures the system according to von Neumann's ideal measurement scheme as usual. This procedure, ending with the coupled environment-system states, is then given by

$$\begin{aligned} |S\rangle |E_0\rangle &= \left(\sum_i s_i |s_i\rangle \right) |\vec{o}_0\rangle \\ &\longrightarrow \sum_i s_i |s_i\rangle |\vec{o}_i\rangle \\ &= \sum_i s_i |s_i\rangle |o_{i_1}, o_{i_2}, \cdots, o_{i_r}, \cdots, o_{i_R}\rangle , \end{aligned} \tag{3.4.1}$$

with the overlap of the final environmental states still tending to negligible in the same

manner as described above in traditional decoherence, $\langle \vec{o}_i | \vec{o}_j \rangle \rightarrow \delta_{ij}$. For sufficiently large environmental fragments, which is assumed to be the case, we can also extend this behaviour to the individual states of the environmental fragments, $\langle o_{i_r} | o_{j_r} \rangle \rightarrow \delta_{ij}$ [135, 31, 134].

In this final form it is easy to see that an observer that can only measure the state of their local environmental fragment, $\mathcal{F}_r$, will be able to obtain the state of the system of interest without ever actually measuring it. This can be done by R individual observers that each measure their local fragment and receive the same consistent result. For this to be the case the information about the state of the system must be stored in the environment with a minimum redundancy of R, one copy of the information for each possible observer that might measure their fragment. The process of the environment measuring the system still results in environmental decoherence suppressing the interference terms of the reduced density matrix of the system, and by extension, the observer's local fragment. As a result, the only information to survive interaction with the environment to be stored redundantly in the many fragments of the environment is that which is described in the einselected preferred basis, which here is $\hat{S}$. This is the process of Quantum Darwinism. In order to obtain information in any other basis, the observer would be required to measure the entirety of the environment, not just their local disjoint fragment. As such there could hypothetically be many observers who each measure there local fragment and will unintentionally, with no prior knowledge about the system, all gain information in the same basis, allowing them to reach consensus on the result. In this way, any measurement made on these correlated fragments of the environment could be said to be objective according to the definition of objectivity in section 2.3.2. If this is so, then such a measurement must also describe some "element of physical reality" according to EPR [92].

This argument hinges on there being many records of the state of the system stored within the environment. The quantity of these records was referred to as the redundancy with which this information is stored. Above we have seen that, in the situation described there, information about the system in the basis of the observable $\hat{S}$ is stored R times, one per fragment. In this vein the redundancy $R_\delta(\hat{A})$ with which partial ($\delta \neq 0$) information about the system of interest, in the basis of an arbitrary system observable of interest $\hat{A}$, is stored in the environment is defined as the maximum number of disjoint fragments that the

environment can be partitioned into such that each fragment contains a near-perfect (perfect less some tiny fraction $\delta \ll 1$) record of the information about the state of the system in the eigenbasis of some observable of interest $\hat{A}$,

$$R_\delta(\hat{A}) = \max_{\{\mathcal{F}_i\}}\{R : I_{\mathcal{F}_i}(\hat{A}) \geq (1-\delta)I_E(\hat{A}), \forall i = 1, \cdots, R\}. \tag{3.4.2}$$

Here $I_{\mathcal{F}_i}(\hat{A})$ describes the maximum information about the system in the eigenbasis of the arbitrary observable $\hat{A}$ of interest that can be gained by measuring the fragment $\mathcal{F}_i$. $I_E(\hat{A})$ is then the information contained in the entire environment E. The maximization process in finding $R_\delta(\hat{A})$ is done over all possible partitions of environmental subsystems into disjoint fragments [92].

From this definition we see that the average size of a fragment in the most ideal configuration of subsystem partitions is $N/R_\delta(\hat{A})$ where we can define a redundancy ratio as $\mathcal{R}_\delta(\hat{A}) = 1/R_\delta(\hat{A})$, the fraction of the environment that contains a single copy of the information about the system. One could imagine that a random observer measuring on a random partition of subsystems is more likely to obtain information stored in a smaller fraction of the environment. If an observer is to only interact with a random partition of the environment, then that partition is more likely to include more of the information that occupies a smaller fraction of the environment [135].

3.4.2 Quantifying Information

In order to quantify the information used above, $I_{\mathcal{F}_i}(\hat{A})$, we must briefly define some quantities. To do so in a generalized manner, we introduce some complex system given by the Hilbert space $\mathcal{H}_{AX}$ with density matrix states ρ, we define two arbitrary observables $\hat{A} = \sum_i a_i\hat{A}_i$ and $\hat{X} = \sum_i x_i\hat{X}_i$ with projection operators $\hat{A}_i = |a_i\rangle\langle a_i|$ and $\hat{X}_i = |x_i\rangle\langle x_i|$. Each observable acts on its own disjoint sub-system A and X of the total Hilbert space $\mathcal{H}_{AX}$ and whose states are given by the partial density matrices $\rho_A = \mathrm{Tr}_X\{\rho\}$ and $\rho_X = \mathrm{Tr}_A\{\rho\}$ respectively. It then follows from the disjoint nature of the systems on which they act that $\left[\hat{A}, \hat{X}\right] = 0$.

As per Born's rule, we then have that the probability of measuring the eigenvalue a_i of the observable $\hat{A}$ is given by $P(a_i) = \mathrm{Tr}\left\{\hat{A}_i\rho\right\}$, upon which the state changes to reflect this

measurement to $\rho_{|a_i} = \hat{A}_i \rho \hat{A}_i / P(a_i)$. This is also the case for $\hat{X}$.

The first quantity to be defined is a measure of an observer's ignorance about some measurement outcome. We can quantify this ignorance an observer might have about a possible measurement outcome of observable $\hat{A}$ by the entropy $H(\hat{A})$ of the state. There are two commonly used measures of entropy [135, 94], namely Shannon entropy [92, 93],

$$H_S(\hat{A}) = -\sum_i P(a_i) \ln P(a_i), \tag{3.4.3}$$

or von Neumann entropy [31, 30],

$$H_N(\hat{A}) = -\operatorname{Tr}\{\rho_A \ln \rho_A\}. \tag{3.4.4}$$

While the definition of von Neumann entropy does not depend on the exact observable, $\hat{A}$, as is the case for Shannon entropy, we keep the notation $H_N(\hat{A})$ to maintain consistency. It should be understood that the label $\hat{A}$ instead refers to the system on which the observable acts, which in this case is the system A. The difference between these definitions is essentially that Shannon entropy describes the classical entropy of the system while von Neumann entropy includes the quantum nature of the system [109, 135]. The distinction that comes from making use of one of these definitions of entropy over the other is looked at more closely when we discuss the concept of discord in section 3.4.3 below. When it is not necessary to distinguish them entropy will be given simply as $H(\hat{A})$.

In general a measurement of one of the observables $\hat{A}$ or $\hat{X}$ will affect the outcomes of any later measurement of the other. As such we can find a conditional entropy $H(\hat{A}|x_i)$ of an observable $\hat{A}$ on condition the system has already had the observable $\hat{X}$ measured and obtained the result x_i. This conditional entropy is defined as above in equations (3.4.3), with probabilities $P(a_i)$ being replaced with conditional probabilities $P(a_i|x_j) = \operatorname{Tr}\left\{\hat{A}_i \rho_{|x_j}\right\}$, or (3.4.4) with the density matrix ρ_A being replaced with the density matrix after measurement $\rho_{A|x_i}$. This idea of conditional entropy can be further generalized to the case of the observable $\hat{X}$ being measured but no specific outcome being specified. In this case, the average over the possible measurement outcomes is used,

$$H(\hat{A}|\hat{X}) = \sum_i P(x_i) H(\hat{A}|x_i). \tag{3.4.5}$$

We can similarly define a joint entropy $H(\hat{A}, \hat{X})$ that describes our lack of knowledge after measuring both the observables $\hat{A}$ and $\hat{X}$. Since they act on disjoint sub-spaces and therefore commute, the order with which they are measured does not matter. This quantity is also defined according to the definitions for the entropy in equations (3.4.3), with probabilities $P(a_i)$ now being replaced with joint probability $P(a_i, x_j) = \mathrm{Tr}\Big\{\hat{A}_i\hat{X}_j\rho\hat{X}_j\Big\} = \mathrm{Tr}\Big\{\hat{X}_j\hat{A}_i\rho\hat{A}_i\Big\}$, or (3.4.4) with the density matrix ρ_A being replaced with the density matrix $\rho_{|a_i,X_j}$.

Having the means to quantify the information that is unknown, we can define the mutual information known between two systems. This is then the sum of our lacking knowledge about each system individually, less our lack of knowledge after jointly measuring both systems,

$$I(\hat{A} : \hat{X}) = H(\hat{A}) + H(\hat{X}) - H(\hat{A}, \hat{X}). \tag{3.4.6}$$

In the classical case where Shannon entropy is used to find the *classical mutual information*, the definition is found to be equivalent to that which makes use of conditional entropy,

$$I(\hat{A} : \hat{X}) = H_S(\hat{A}) - H_S(\hat{A}|\hat{X}) = H_S(\hat{X}) - H_S(\hat{X}|\hat{A}), \tag{3.4.7}$$

which describes the amount of information about one system that can be obtained by measuring the other. This is differentiated from *quantum mutual information*, $\mathcal{I}(\hat{A} : \hat{X})$, which instead describes the information that each system has of the other [135].

The maximum information that can then be obtained about a system A via a measurement on an arbitrary disjoint system X, which was given as $I_X(\hat{A})$, can finally be defined. In the case of maximum quantum mutual information we rather straightforwardly have that $\mathcal{I}_X(\hat{A}) = \mathcal{I}(\hat{A}, \hat{X})$. This follows from von Neumann entropy not depending on the observable that is being measured, only the systems states. For the observable dependant classical mutual information, it is required to find the observable defined on the measured system X that gives the maximum amount of information about the system A in the eigenbasis of some observable $\hat{A}$,

$$I_X(\hat{A}) = \max_{\{\hat{X}\}} I(\hat{A} : \hat{X}). \tag{3.4.8}$$

In this classical case, the maximum information that could theoretically be obtained is $I(\hat{A} : \hat{X}) \leq H_S(\hat{A})$, the information about A alone. This is in contrast to quantum mutual information that takes into account the possible entanglement between the systems, allow-

ing for information about both states to be gained simultaneously, resulting in a theoretical maximum quantum mutual information of $\mathcal{I}(\hat{A} : \hat{X}) \leq H_N(\hat{A}) + H_N(\hat{X})$.

Applying this construction to the situation given by equation (3.4.1), we can associate the systems A and X with the system of interest S and an environmental fragment $\mathcal{F}_i$ respectively. By performing appropriate partial traces, the density matrices ρ_S and $\rho_{\mathcal{F}_i}$ can also be obtained. This allows us to quantify the maximum information that can be obtained about the system of interest S via measurements on the fragment $\mathcal{F}_i$. In the case where the environmental fragment is chosen to be the entire environment, we can also find the maximum information about the system in the basis of some observable $\hat{A}$ stored in the environment, $I_E(\hat{A})$. This is a rather important quantity as only when there is enough information in the environment to compensate for the observer's initial lack of knowledge can we gain any insight about $\hat{A}$ by measuring parts of the environment E, i.e. that $I_E(\hat{A}) \approx H(\hat{A})$. With these quantities all quantified, one can begin to investigate the results of the redundancy $R_\delta(\hat{A})$ with which information about the system in the basis of the observable $\hat{A}$ is stored in the environment, as given by equation (3.4.2).

3.4.3 Discord and the Different Informations

We have defined two different types of mutual information, one that relates to the quantum nature of the system, $\mathcal{I}(\hat{A} : \hat{X})$, and another to the classical, $I(\hat{A} : \hat{X})$. Quantum mutual information, defined with the use of von Neumann entropy H_N is a measure of how strongly correlated the sub-system A and X are. Specifically, it is the amount of entropy, and hence uncertainty, that might be gained if the correlations between the two systems are destroyed. Classical mutual information $I(\hat{A} : \hat{X})$, defined with Shannon entropy H_S, is then a measure of our ability to predict, in the classical sense, the outcome of measurements of a given observable $\hat{A}$ on the sub-system A given prior measurements of observable $\hat{X}$ on the other sub-system X. Classical mutual information is the information known by the observer about the system A while quantum information is the information the systems know of each other [94, 135].

The dependence of classical mutual information, and Shannon entropy specifically, on the measured observable shows how the information that an observer might obtain about the

system is dependent on said observable. Contrast this to the lack of any dependence on a measured observable in the definition of the quantum mutual information, which takes into account quantum entanglement of the systems. This extra information about the quantum correlations within the systems would then seem to, in general, not be obtainable by an observer unless the correct observable is measured. In this way we have that $I(\hat{A} : \hat{X}) \leq \mathcal{I}(\hat{A} : \hat{X})$ [94, 135].

The discord can then be defined as a means of quantifying this information unobtainable to the observer,

$$\Delta I(\hat{A} : \hat{X}) = \mathcal{I}(\hat{A} : \hat{X}) - I(\hat{A} : \hat{X}). \tag{3.4.9}$$

This is then a positive valued quantity which measures the efficiency with which a measurement of the observable $\hat{X}$ on the system X obtains information about the observable $\hat{A}$ on the system A [135]. It describes the information about the system A that is inaccessible to the observer via measurement on system X, and as such representative of the quantum nature of the correlations between the systems [94, 135].

In the case of a coupled system-environment state post decoherence, we see that measurements in the pointer basis of the environment then minimize the discord. This then allows discord to become a measure of a proposed basis as a choice of pointer basis as well as a means of determining the efficiency of einselection [94, 135]. When measurements are made in the pointer basis the discord is minimized and any differences that might come from using one definition of mutual information over the other is lost. As such, the redundancy that is found using either definition returns similar results. Typically, in the case of macroscopic systems, the redundancy is of such high order that the differences caused by your choice of mutual information are not significant [135, 107].

3.4.4 Nature of the Available Information

The information that is most likely to be obtained via a measurement on a random environmental fragment is that information that is most redundantly stored in the environment. It is then natural to ask after the nature of such information [92, 93, 135]. What sort of information about the system is stored most redundantly, and as such can be found most

easily by probing the environment? The act of gaining this information, however, includes measuring some observable $\hat{X}$ on a chosen fragment of the environment $\mathcal{F}$. As a result, the information that is gained is classical in nature. For this reason, the classical quantities defined using Shannon Entropy will be used going forward until mentioned otherwise.

Here we shall make use of the case leading to equation (3.4.1). We note here that the construction equation is of perfect correlation between outcomes of observables $\hat{S}$ and $\hat{O}$, with eigenvalue equations $\hat{S}\left|s_i\right\rangle = S_i\left|s_i\right\rangle$ and $\hat{O}\left|o_i\right\rangle = O_i\left|o_i\right\rangle$, acting on the disjoint system S and fragment $\mathcal{F}$ respectively. We refer to these observables as the interaction observables. We have dropped the subscript that identifies the fragment in relation to other fragments defined on the environment for the duration of this section as it is unnecessary, but it will be reintroduced in the future. Arbitrary observables $\hat{A}$ and $\hat{X}$ defined on the disjoint system S and fragment $\mathcal{F}$ respectively, having zero commutation relation $[\hat{A}, \hat{X}] = 0$, will then have positive mutual information $I(\hat{A} : \hat{X})$, implying that $H_S(\hat{A}|\hat{X}) \leq H_S(\hat{A})$.

In order to discuss the nature of information with the largest redundancy, as given by equation (3.4.2), one must first find the observable $\hat{F}$ that would give the largest amount of information about the system of interest S upon a measurement on a fixed random fragment $\mathcal{F}$,

$$I_{\mathcal{F}}(\hat{A}) = I(\hat{A} : \hat{F}) \geq I(\hat{A} : \hat{X}), \tag{3.4.10}$$

for all observables $\hat{X}$ defined on the fragment $\mathcal{F}$, given some observable $\hat{A}$ defined on the system S. Here we make use of the definition of maximum information, equation (3.4.8), to connect it to the required observable $\hat{F}$. Finding this observable requires maximizing the mutual information, which in accordance with equation (3.4.7), is equivalent to minimizing the conditional entropy $H_S(\hat{A}|\hat{X})$.

To this end, we describe the combined system-fragment state $|S, \mathcal{F}\rangle$ in the basis of observables $\hat{A}$ for the system, with eigenvectors $\{|a_j\rangle\}$, and an arbitrary observable $\hat{X}$ which acts on the environmental fragment, with eigenvectors $\{|x_k\rangle\}$. As such, we describe the system-fragment state as

$$|S, \mathcal{F}\rangle = \sum_i s_i \left|s_i\right\rangle \left|o_i\right\rangle = \sum_{i,j,k} s_i \left\langle a_j|s_i\right\rangle \left\langle x_k|o_i\right\rangle \left|a_j\right\rangle \left|x_k\right\rangle . \tag{3.4.11}$$

In attempting to minimize the conditional entropy $H_S(\hat{A}|\hat{X})$, we then investigate the effect

of the quantity $\langle x_k|o_i\rangle$ as this is what will differ with various choices of $\hat{X}$. In solving for the conditional entropy, we make use of the quantities

$$P(x_k) = \sum_i |s_i|^2 |\langle x_k|o_i\rangle|^2, \tag{3.4.12}$$

$$P(x_k)P(a_j|x_k) = \sum_{i,i'} s_i s_{i'}^* \langle s_{i'}|a_j\rangle \langle a_j|s_i\rangle \langle o_{i'}|x_k\rangle \langle x_k|o_i\rangle, \tag{3.4.13}$$

found from the Born's rule, $P(x_k) = \mathrm{Tr}\{\langle x_k|S,\mathcal{F}\rangle \langle S,\mathcal{F}|x_k\rangle\}$. The conditional entropy, written in full as $H_S(\hat{A}|\hat{X}) = -\sum_{jk} P(x_k)P(a_j|x_k)\ln P(a_j|x_k)$, can then be seen to depend on the overlap $\langle x_k|o_i\rangle$ between the eigenbasis of the arbitrary observable $\hat{X}$ and the interaction observable $\hat{O}$. There are three cases of interest in this investigation, namely

1. $\langle x_k|o_i\rangle = c \quad \forall i,j$ where c is some constant,
2. the special case where $\langle x_k|o_i\rangle$ are chosen such that $\sum_{j,k} \langle a_j|s_i\rangle \langle x_k|o_i\rangle |a_j\rangle |x_k\rangle = \alpha_i |a_i\rangle |x_i\rangle$,
3. and $\langle x_k|o_i\rangle = \delta_{k,i}$.

Beginning with the first case, this results in maximum conditional entropy and would result in no insight into the nature of the state of the system, reducing our uncertainty about the state of the system by nothing, $H(\hat{A}|\hat{X}) = H(\hat{A})$. The second case offers a situation in which we have perfect correlation between the system observable $\hat{A}$ and the arbitrary environmental fragment observable $\hat{X}$. Here one would expect perfect information about the system, but the process of decoherence drives the interference in the eigenbasis of the interaction observable only to be suppressed, $\langle o_i|o_j\rangle \to \delta_{ij}$, as such a measurement of the arbitrary observable $\hat{X}$ on the environmental fragment would not include any information about quantum correlations. It is only the third case, that of observables that are aligned with the interaction observable, that gives maximal information about the system, $\hat{F} = \hat{O}$. This is not a very surprising conclusion based on the forms of equations (3.4.12) and (3.4.13). As such we find that those observables most aligned with the interaction observable are the ones that minimize the conditional entropy and the maximize mutual information,

$$I(\hat{A}:\hat{O}) \geq I(\hat{A}:\hat{X}) \tag{3.4.14}$$

for all $\hat{X}$ defined on the environmental fragment $\mathcal{F}$ [92]. This procedure is reminiscent of the predictability sieve procedure used to identify the preferred basis that results from

einselection, showing how the emergence of a preferred basis in which one would make measurements on their local environmental fragment. Similar arguments to those made here can be used to find the system observable $\hat{C}$ that a measurement of an arbitrary observable $\hat{X}$ on the environmental fragment would give the largest amount of information about, $I(\hat{C} : \hat{X}) \geq I(\hat{A} : \hat{X})$ for all observables $\hat{A}$ defined on the system. In the same vein as the above result, we find that measurements made on arbitrary observables of the fragment obtain greatest information about the original interaction observable of the system $\hat{S}$, i.e.

$$I(\hat{S} : \hat{X}) \geq I(\hat{A} : \hat{X}) \tag{3.4.15}$$

for all observables $\hat{A}$ defined on the system [92]. From these results we see that if one is attempting to gain maximum information about the state of the system, they are incentivized to choose the interaction basis $\hat{O}$ to perform their measurements on their local fragment, and in doing so, gaining maximal information of the interaction basis of the system $\hat{S}$ [92, 93, 135].

Naturally, the mutual information obtained via measurement of the interaction observables gives complete information about the state of the system in the case of perfectly correlated system-environment states, $I(\hat{S} : \hat{O}) = I(\hat{S})$. As a result of this perfect correlation between the interaction observables $\hat{S}$ and $\hat{O}$, information obtained by measurements of the observable $\hat{O}$ is equivalent to that obtained from measuring the system directly, $I(\hat{A} : \hat{O}) = I(\hat{A} : \hat{S})$. We can make use of this in equation (3.4.14) above and equation (3.4.8) to find the maximum information that can be obtained about an arbitrary observable $\hat{A}$ via measurements performed on the environmental fragment $\mathcal{F}$,

$$I_{\mathcal{F}}(\hat{A}) = I(\hat{A} : \hat{F}) = I(\hat{A} : \hat{S}). \tag{3.4.16}$$

Here we see that the maximum information about the state of the system in the basis of our arbitrary observable $\hat{A}$ that can be obtained by a measurement on the environmental fragment $\mathcal{F}$ is inherently limited by the mutual information between said observable and the interaction observable of the system. It would seem that we can only gain any significant information about observables that are aligned with the original interaction observables [92, 93, 135].

This result also allows us to bypass any dependence on the exact construction of the environmental fragments by noting that the maximum information about this observable of interest $\hat{A}$ can be found by making use only of its relationship to the original interaction observable $\hat{S}$ which has no dependence on the exact construction of the environmental fragments. This allows us to make statements that are true for any construction of the fragments for which the fragment is not equal to the entirety of the environment, $E \setminus \mathcal{F} \neq \emptyset$. We then have that the redundancy of the information about the observable of interest $\hat{A}$ is completely dependent on its alignment with the interaction observable of the system $\hat{S}$ [92, 93, 135].

This becomes immediately useful for more directly making statements about the redundancy with which information on our observable of interest $\hat{A}$ is stored in the environment. In order to find the redundancy with which information is stored in the environment, equation (3.4.2), we compare the maximum information that can be obtained from a single fragment $I_{\mathcal{F}}(\hat{A}) = I(\hat{A} : \hat{S})$ to that which is stored in the near ($\delta \ll 1$) entirety of the environment $(1-\delta)I_E(\hat{A})$. We then have two situations: The first being that the information that can be obtained about the observable of interest $\hat{A}$ via measurement on a fragment of the environment is not sufficient, $I(\hat{A} : \hat{S}) < (1-\delta)I_E(\hat{A})$. This situation amounts to there being no means of constructing an environmental fragment that contains the full information about the observable of interest $\hat{A}$. This means that only a measurement on the entirety of the environment will return the complete information about the observable $\hat{A}$, and as such the redundancy is then $R_\delta(\hat{A}) = 1$ [92, 93, 135, 31]. The second situation is that of sufficient information, $I(\hat{A} : \hat{S}) \geq (1-\delta)I_E(\hat{A})$. This inequality then states that sufficient information about $\hat{A}$ can be derived from information about the interaction observable $\hat{S}$, as such any fragment that contains information about $\hat{S}$ also contains sufficient information about $\hat{A}$. As such we have that the redundancy with which near-perfect ($\delta > 0$) information about the system observable $\hat{A}$ is stored in the environment is given by $R_\delta(\hat{A}) \geq R_{\delta=0}(\hat{S})$. As the quality of the information increases $\delta \longrightarrow 0$ for which $I(\hat{A} : \hat{S}) \geq (1-\delta)I_E(\hat{A})$ is still true, $R_\delta(\hat{A}) \geq R_{\delta=0}(\hat{S})$ would tend to equality. If the environment is large, as it traditionally is assumed to be in the decoherence scheme, and we have as a result the redundancy of the interaction observable being very large $R_\delta(\hat{S}) \gg 1$, then these results suggest only two rather extreme options for the redundancy of information about arbitrary observables $\hat{A}$. Either

one would have to measure the entirety of the environment in order to learn anything about the system, an impossible task, or, as is the case for those observables closely aligned with the interaction observables, the redundancy of the observable is comparable to the already stated large magnitude of the redundancy of the interaction observable $\hat{S}$. Either information about an observable of the system is impossible to come by or it is very common [31, 30]. Additionally, repeating the results from above, it is those observables that are aligned with the interaction observable that have obtainable information stored within the environment. As such we see that the dynamical process of decoherence selects for a very particular set of observables to be distributed throughout the environment in a manner reminiscent of survival of the fittest [92, 93], hence Quantum Darwinism, with any competing observables being effectively unobtainable. In this way, there exists a readily and redundantly stored information in the environment which can be obtained by various observers without measuring the system directly, and hence disturbing it, and the results obtained by such measurements will be consistent with one another. This shows how the process of Quantum Darwinism results in the emergence of objectivity as given in section 2.3.2, and as such contributes to the transition from quantum to classical [31, 30, 135].

These observables that are selected by the environment can also be seen to minimize the discord of the system-fragment state. This not only reinforces that the redundancy used here is comparable to that which might be found using quantum mutual information, but also suggests that the discord can be used to help identify einselected observables. This also allows discord to be used as a measure for objectivity since it is those observables that minimize discord that have been found to be most objective [94, 135, 136].

Above we only looked at the case of perfect correlation between the states of the observables $\hat{S}$ and $\hat{O}$. Situations with such perfect correlations are, however, not very common in nature and the discussion above is very much an idealization. In general, we will not have that the states of the environmental fragment will so uniquely describe the state of the system and we might have that $\langle o_i|o_j\rangle \neq \delta_{ij}$. Rather we might have only near correlation in which $|\langle o_i|o_j\rangle| \ll 1$. In these more realistic cases, we still see that the above results apply and that the mutual information between two observables $\hat{A}$ and $\hat{X}$ for the imperfect case $\tilde{I}(\hat{A}:\hat{X})$ is still approximately that of the perfect case, $\tilde{I}(\hat{A}:\hat{X}) \approx I(\hat{A}:\hat{X})$. This is true as long as we

have that the environmental states $|\langle\vec{o}_i|\vec{o}_j\rangle| \ll 1$ and that both the size (in terms of number of environmental subsystems) of the fragment $\mathcal{F}$ and the remaining environment less the fragment, $E \setminus \mathcal{F}$, scale as $\ln(\dim(S))$ where $\dim(S)$ is the dimension of the system of interest S [92, 135]. The accuracy with which the results found above in the case of perfect correlations apply to the general case of near correlation is then roughly $\ln(\dim(S))/N$ where N is the number of environmental subsystems. In the case of most macroscopic scenarios with typically very many environmental subsystems $N \gg 1$, this is a rather acceptable accuracy [92].

4. The Dynamical Reduction Program

Models of Dynamical Reduction, often referred to as just Collapse models, offer a means of describing the transition from quantum to classical dynamics by assuming instead both are approximations of some deeper description. It proposes that one cannot observe quantum behaviour in the classical regime because you are applying quantum descriptions outside the range that it, as an approximation, is valid. This regime also does not include the interaction of the system with the apparatus, and hence quantum dynamics cannot be used to describe the measurement process in which the state of the system, initially in a superposition of eigenstates of the observable measured by the apparatus, collapses to a single eigenstate. Collapse is assumed to be a dynamical, physical process and collapse models attempt to provide a description of these dynamics, bridging the descriptions of classical and quantum mechanics and speaks to some underlying universal description.

To show this transition, we start with some system whose dynamics are described by a Hamiltonian H. If it were a classical system, it would be described by a set of particles with position and momentum vectors, s_i and p_i respectively, in phase space (s_i, p_i). In the case of a quantum description, the system would be described by some state vector $|s\rangle$ or wave-function ψ_s in a Hilbert space $\mathcal{H}_s$ where position and momentum are operators $\hat{s}_i$ and $\hat{p}_i$ with commutation relation $[\hat{s}_i, \hat{p}_j] = i\hbar\delta_{ij}$. In classical mechanics the dynamics of the system can be described via the use of the Hamilton-Jacobi equation,

$$\frac{\partial S}{\partial t} = -H(s_i, \frac{\partial S}{\partial s_i}), \tag{4.0.1}$$

where S is the action of the system, with $p_i = \frac{\partial S}{\partial s_i}$. In quantum mechanics by contrast the wave function of the system evolves in time according to the Schrödinger equation,

$$i\hbar\frac{\partial \psi}{\partial t} = H\psi, \tag{4.0.2}$$

with H the standard non-relativistic Hamiltonian, $H = \frac{-\hbar^2}{2m}\nabla^2 + V(\hat{x})$. It is insightful to consider here a single particle of mass m in one dimension and compare the classical and quantum descriptions of its evolution in time. To do so we define $\psi = e^{iS/\hbar}$ and write the

Schrödinger equation in the position representation as

$$\frac{\partial S}{\partial t} = \frac{-1}{2m}\left(\frac{\partial S}{\partial s}\right)^2 - V(s) + \frac{i\hbar}{2m}\frac{\partial^2 S}{\partial s^2}. \tag{4.0.3}$$

We compare this to the classical case where the particle's dynamics are described by the Hamilton-Jacobi equation, which here is then

$$\frac{\partial S}{\partial t} = \frac{-1}{2m}\left(\frac{\partial S}{\partial s}\right)^2 - V(s). \tag{4.0.4}$$

Here we have assumed the action S to be real-valued for the sake of this discussion. This Hamilton-Jacobi equation is then just the Schrödinger equation, equation (4.0.3) above, under the condition that its last term is neglected, which corresponds to the macroscopic limit $S \gg \hbar$.

The dynamics of the system in some Hilbert space given by the Schrödinger equation can be transitioned to a description in phase space given by the Hamilton-Jacobi equation in this limit. In doing so, however, we lose the linearity of the Schrödinger equation and the phenomenon of superposition that is so characteristic of quantum mechanics and one of the properties that differentiate quantum mechanics from the classical, where such superposition is not ever observed. It can then be argued that the Hamilton-Jacobi equation, and the dynamics described by it, are merely an approximation of some underlying theory that also approximates quantum dynamics described by the Schrödinger equation in the appropriate limits, as this transition seems to suggest.

It is assumed that this underlying theory must be *nonlinear* in order for superposition to break down during measurement, but this non-linearity must be negligible in the microscopic domain in order for it to match what is observed in quantum dynamics. Disregarding the measurement problem, nonlinear modifications of the Schrödinger equation have been considered before due to the lack of linearity of most other descriptions of dynamics in nature. Linearity in these cases is then usually a convenient limiting approximation. Taking inspiration of the probabilistic nature of the measurement process described by Born's rule, collapse models further assume the universal description to be *stochastic* in nature, however, this stochasticity must become negligible in the microscopic limit in order to reproduce the deterministic nature of the Schrödinger equation. Deterministic models of dynamic state reduction

have been studied as well but it is generally found that such models result in superluminal propagation [103, 77]. Additionally, in order for the decay of all but one measured eigenstate, the dynamics proposed by collapse models describe *non-unitary but norm-preserving* evolution of the system state. Once more, this behaviour should become negligible for microscopic systems. For all the above properties, the stochastic nonlinear non-unitary dynamics, there is an amplification mechanism within the models that allows for the dynamics to coincide with the macroscopic descriptions we observe. These properties also ensure position localization for macroscopic systems and as such offer an explanation for the observed lack of macroscopic superposition.

We can see then that the dynamics described by these models for mesoscopic (neither microscopic nor macroscopic) systems is then distinguishable from those described by the Schrödinger equation or the Hamilton-Jacobi equation, and hence can be experimentally distinguished. These descriptions also are not focused on the resolution of the measurement problem *per se* but rather the measurement process happens to be within this mesoscopic regime and as such could be described by these models.

4.1 Collapse Models

In his pioneering paper in 1976 Pearle first attempted to apply a stochastic nonlinear modification of the Schrödinger equation to explain state vector reduction [99]. It was proposed that immediately after interaction with a macroscopic apparatus certain variables take on random values which cause the breaking of superposition resulting in the measurement outcome. Pearle suggested the state vector phases immediately after measurement as the random variables, which when assigned the correct probability distribution would result in Born's rule. The programme, however, did not account for a preferred basis which prevented the model to be perfectly universal.

While this model has been improved on since then, it demonstrates the disconnect still to be resolved between these phenomenological models and underlying theory. The probability distribution of the chosen random variables are still inserted by hand with the required outcome in mind. There have been attempts to connect these distributions to something

more fundamental that might explain them, but the issue cannot be said to be resolved as of yet [26].

4.1.1 GRW Model

The first consistent model to be proposed that overcame the problem of preferred basis and included an amplification mechanism was the model proposed by Ghirardi, Rimini, and Weber (GRW) in [75] which as such became known as the GRW model, or QMSL model for quantum mechanics with spontaneous localization. Most models since have been based on this initial model. For some system of N distinguishable particles, assumed to be spinless and scalar for simplicity, the state of the system at time t is described by the combined wavefunction $\psi(t, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \cdots, \vec{\mathbf{x}}_N)$ in a position representation $\langle \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \cdots, \vec{\mathbf{x}}_N | \psi(t) \rangle$ of the Hilbert space $\mathcal{H}_S$. The GRW model was designed keeping in mind two requirements, namely that

1. any preferred basis must be chosen such that the wavefunction of macroscopic objects reduce to localized positions in space,

2. and there must be some amplification mechanism in place that acts to destroy the superposition in macroscopic systems but is negligible at the microscopic scale.

To achieve that, it is assumed that the system evolves according to the Schrödinger equation for most of the time, however, each particle in the system undergoes a sudden and spontaneous process that acts to localize it according to the first point above. The probability of such a process happening to a single particle n at any given time is assumed to be similar to a Poisson process with frequency λ_{GRW} and is distributed in space according to the distribution

$$p_n(\vec{\mathbf{x}}) \equiv \| \hat{L}_n(\vec{\mathbf{x}}) \psi(t, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \cdots, \vec{\mathbf{x}}_N) \|^2, \tag{4.1.1}$$

with the probability of this process happening being independent between different particles. The probability distribution with which the location of localization is chosen aligns with the frequency with which one expects to measure the position of the particle in standard quantum mechanics according to Born's rule. Here we have that the norm–reducing, positive,

self–adjoint, linear Localization operator $\hat{L}_n(\vec{\mathbf{x}})$ at point $\vec{\mathbf{x}}$ is then given by

$$\hat{L}_n(\vec{\mathbf{x}}) = \frac{e^{-(\hat{x}_n - \vec{\mathbf{x}})^2/2r_c^2}}{(\pi r_c^2)^{3/4}}, \tag{4.1.2}$$

with $\hat{x}_n$ the position operator of the n-th particle in the system, r_c a parameter that determines the width of the localization process, and $\vec{\mathbf{x}}$ the random variable. Upon the stochastic act of localization of particle n, the wavefunction of the system undergoes a sudden jump of the form

$$\psi(t, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \cdots, \vec{\mathbf{x}}_N) \to \frac{\hat{L}_n(\vec{\mathbf{x}})\psi(t, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \cdots, \vec{\mathbf{x}}_N)}{\|\hat{L}_n(\vec{\mathbf{x}})\psi(t, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \cdots, \vec{\mathbf{x}}_N)\|}. \tag{4.1.3}$$

At all other times the system evolves according to the Schrödinger equation.

In the above formulation, two new quantities were introduced that parameterize the localization process and are taken as constants of nature. Namely, these quantities are r_c, which describes the width of the localization process, and λ_{GRW}, which is the average localization frequency of the Poisson process which describes how often localization takes place. The exact values for these quantities have yet to be found exactly, but based on existing investigations they are taken to be of the order

$$\lambda_{GRW} \simeq 10^{-16}\ \mathrm{s}^{-1}, \quad r_c \simeq 10^{-7}\ \mathrm{m}. \tag{4.1.4}$$

This lack of exact values for what are supposed to be constants of nature speaks to the disconnect between these models and any underlying physical theory. Efforts are being made to find these values to higher degrees of accuracy. In the meantime, these values have been used to show that the dynamics described by the GRW model reproduce, as far as current technologies allow us to measure, both microscopic dynamics, in accord with that described by the Schrödinger equation, as well as macroscopic dynamics [22]. These modifications of the standard quantum mechanics also reproduce the measurement procedure, namely the Born probability rule and wave packet reduction. With this, the GRW model of instantaneous collapse offers a unified description of physical dynamics at all scales as well as a description of the process of measurement.

Here we note the lack of any particles in this model, in the sense of a delta-function like wavefunction. Instead the n-th "particle" of a system is described by a distribution of mass

$m_n(t, \vec{\mathbf{x}}_n)$ at time t according to

$$m_n(t, \vec{\mathbf{x}}_n) \equiv m_n \int d^3x_1 \cdots d^3x_{n-1} d^3x_{n+1} \cdots d^3x_N |\psi(t, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \cdots, \vec{\mathbf{x}}_N)|^2, \tag{4.1.5}$$

where m_n is the mass of the n-th particle. While this can imply for microscopic systems that the mass is spread out in space and hence the description of "particle" is not fitting, we shall continue the use of this terminology regardless. This generally spread out nature of particles according to this model means that for an electron in the double split experiment, the electron does indeed physically diffuse through both slits in a manner reminiscent of a classical wave. The quantum nature of the electron then appears upon attempting to locate this spread out medium in space upon interaction with a well-localized macroscopic photographic film. This interaction then includes the rapid shrinking of this electron to the final point that is then measured as its final outcome position. This process of measurement is added *ad hoc* onto the standard formulation of quantum mechanics, but is a natural result of those dynamics described by the GRW model. As a result of this property of the GRW model, it so happens that when many particles interact, they cause each other to be very well localized in space, contributing to what we think of as a property of macroscopic objects.

While the discontinuous nature of the sudden localization proposed in the GRW model means we cannot describe the dynamics of the wavefunction in a single modified Schrödinger equation, we can, however, offer a description of the collapse dynamics of the density matrix $\rho(t) = |\psi(t)\rangle \langle\psi(t)|$ of the system via the master equation for the GRW model [22, 75]

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\frac{i}{\hbar}[H, \rho(t)] - \lambda_{GRW} \sum_{i}^{N} (\rho(t) - T_i[\rho(t)]), \tag{4.1.6}$$

where H is the standard total Hamiltonian of the system, and the last term including $T_i[\rho(t)]$ is the effect of the reduction on the wavefunction of each particle i in the system where $T_i[\rho(t)]$ is given by

$$T_i[\rho(t)] = \int \mathrm{d}\vec{\mathbf{x}}\, e^{-(\hat{x}_i - \vec{\mathbf{x}})^2/2r_c^2} \rho(t) e^{-(\hat{x}_i - \vec{\mathbf{x}})^2/2r_c^2}, \tag{4.1.7}$$

where $\hat{x}_i$ is the position operator of the i-th particle.

Separating the description of the dynamics of the system as a whole from the internal dynamics via identifying centre of mass and relative co-ordinate operators, $\hat{X}$ and $\hat{r}$ respectively, we can identify the relative behaviours of each. Here the Hamiltonian is given by

$H = H_X + H_r$, with H_X and H_r the partial Hamiltonian of the centre of mass and relative co-ordinates respectively. It can be shown [75] that the master equation, equation (4.1.6), can be separated into the master equation of centre of mass, $\rho_X(t)$, and relative co-ordinates, $\rho_r(t)$, alone via appropriate partial traces of the system. These are then given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_r(t) = -\frac{i}{\hbar}[H_r, \rho_r(t)], \quad \text{and} \tag{4.1.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_X(t) = -\frac{i}{\hbar}[H_X, \rho_X(t)] - N\lambda_{GRW}(\rho_X(t) - T_i[\rho_X(t)]). \tag{4.1.9}$$

These results show us that the internal dynamics of a system are still subject to standard quantum mechanics, equation (4.1.8), while the dynamics of the centre of mass behave as if they are a single particle described by the original master equation above, equation (4.1.6), but with a new collapse rate, $N\lambda_{GRW}$, that scales with the size of the system.

In the case of a single particle such as that described by the centre of mass master equation, investigating the final term of the density matrix master equation, equation (4.1.6) above, in the position representation we have

$$\langle\vec{\mathbf{x}}|\, \lambda_{GRW}\left(\rho(t) - T[\rho(t)]\right) |\vec{\mathbf{y}}\rangle = \lambda_{GRW}\left(1 - e^{-(\vec{\mathbf{x}}-\vec{\mathbf{y}})^2/4r_c^2}\right) \langle\vec{\mathbf{x}}|\, \rho(t)\, |\vec{\mathbf{y}}\rangle\,, \tag{4.1.10}$$

demonstrating the suppression of the off-diagonal terms of the density matrix at a rate proportional to λ_{GRW} and dependant on the relative "distance" from the diagonal, the superposition of states closer together lasting longer, driving the system towards localization. This applied to the centre of mass of some large system shows the amplification mechanism of the GRW model at work. Here we see that suppression of off-diagonal terms is proportional to the size of the system via the increased rate of collapse for larger systems, $N\lambda_{GRW}$.

This N dependent amplification process, with N being the number of particles in the object, can be demonstrated differently by considering the localization process happening to a single one of the N particles in a macroscopic object whose state is in a superposition of macroscopically distinguishable positions. For such a macroscopic object, each of its constituent particles must also be in some macroscopic superposition of position states. If a single particle were then to undergo a localization process, it would then be localized to one or the other positions, forcing the system into a statistical mixture of the position states instead of a superposition. Since there are N particles making up this macroscopic object, the rate

with which this process occurs must scale with N. This demonstrates how the model offers a description of both macroscopic as well as microscopic dynamics simultaneously within a single unified principle by explaining macroscopic properties as emergent from underlying microscopic processes.

The GRW model as presented does however have some caveats. In the form introduced here, it can only be used to describe a set of distinguishable particles. Generalizations to the case of identical particles have been found via the use of particle positions or local densities to identify configurations, but they have features that are not ideal or just not as good as other models, as we will see below. The issue stems from the fact that any generalization to a system of identical particles must compete with the destruction of any symmetry properties in the state vector suffered during the localization process. The next issue is an aesthetic one: the lack of a concise single differential equation to describe the dynamics of the system. The result has the feel of "hard-coding" the collapse postulate as a stochastic process, with probability governed by Born's rule, into the dynamics of the system with extra rules to hold it in place, all to avoid calling the collapse a postulate. Regardless of its lack of concise description, the dynamics are still perfectly defined.

4.1.2 CSL model

Based on the ideas of the GRW model, a new model was developed that resolved the above difficulties and is the most advanced model developed so far [98, 74]. The new model, referred to as the *continuous spontaneous localization* (CSL) model, modifies the Schrödinger equation with a set of real Wiener processes $\vec{\mathbf{W}} = \{W_i\}$ that couples with some set of collapse operators $\mathbf{A} = \{\hat{A}_i\}$ chosen according to a "preferred basis" defined by the experimenters. This choice is also subject to the conditions that [22]

- objects of the macroscopic scale are always localized in position space,
- the dynamics of microscopic degrees of freedom have only negligible or smaller alterations to their dynamics such that standard quantum dynamics are still a valid description,

- any breaking of the conservation of energy due to the localization process is at most not detectable,
- systems of many identical particles maintain there symmetry properties.

The CSL model is constructed to be continuous in time, allowing for the dynamics to be described by a single stochastic differential equation in contrast to the dynamics of the GRW model. In this way the description of the CSL model contains both the description of standard quantum mechanics given by the Schrödinger equation, as well as the dynamics that lead to the localization of the state. In its most general form, before any choices for $\mathbf{A}$ are made, the CSL model is described by the stochastic differential equation given in the Itô formalism as

$$\mathrm{d}\left|\psi(t)\right\rangle = \left[\frac{-i}{\hbar}\hat{H}\,\mathrm{d}t + \frac{1}{2}\left((\mathbf{A}-\langle\mathbf{A}\rangle)\cdot\langle\mathbf{A}\rangle - (\mathbf{A}^{\dagger}-\langle\mathbf{A}\rangle)\cdot\mathbf{A}\right)\mathrm{d}t + (\mathbf{A}-\langle\mathbf{A}\rangle)\cdot\mathrm{d}\vec{\mathbf{W}}\right]\left|\psi(t)\right\rangle, \tag{4.1.11}$$

which is nonlinear and preserves the norm, $\|\psi(t)\|^2$ [22, 74, 100, 73]. In the more interesting case in which the collapse operators $\mathbf{A}$ are chosen to be self-adjoint, the above reduces to the simpler form of

$$\mathrm{d}\left|\psi(t)\right\rangle = \left[\frac{-i}{\hbar}\hat{H}\,\mathrm{d}t - \frac{1}{2}(\mathbf{A}-\langle\mathbf{A}\rangle)^2\,\mathrm{d}t + (\mathbf{A}-\langle\mathbf{A}\rangle)\cdot\mathrm{d}\vec{\mathbf{W}}\right]\left|\psi(t)\right\rangle, \tag{4.1.12}$$

where $\hat{H}$ is the standard Hamiltonian, $|\psi(t)\rangle$ the system state vector in some Hilbert space $\mathcal{H}_S$, and $\langle A\rangle$ has been used as short hand for $\langle A\rangle \equiv \frac{1}{2}\langle\psi|\,(\mathbf{A}^{\dagger}+\mathbf{A})\,|\psi\rangle$ (or $\langle A\rangle \equiv \langle\psi|\,\mathbf{A}\,|\psi\rangle$ in the self-adjoint case). The set of real Wiener processes $\vec{\mathbf{W}} = \{W_i\}$ are chosen to have the properties

$$\langle \mathrm{d}W_i\rangle = 0, \quad \text{and} \quad \langle \mathrm{d}W_i\,\mathrm{d}W_j\rangle = \delta_{ij}\,\mathrm{d}t\,. \tag{4.1.13}$$

With the correct choice of collapse observable, the CSL model then overcomes all the problems of previous models and so far is the most advanced fully developed non-relativistic collapse model, capable of making definite predictions that can be applied to any non-relativistic experimental results.

Since the only models that can guarantee a timely localization process for the position of macroscopic objects are those that choose their preferred collapse basis to be position, we will only consider models of these types. The requirement that symmetry properties of the system are maintained leads us to take inspiration from the second quantization of quantum

mechanics. We then consider the creation and annihilation operators $\hat{a}_j^\dagger(\vec{\mathbf{y}})$ and $\hat{a}_j(\vec{\mathbf{y}})$ of particles of type j at point $\vec{\mathbf{y}}$. From these we define the self-adjoint locally averaged density operator

$$\hat{N}_j(\vec{\mathbf{x}}) = \int \mathrm{d}^3\vec{\mathbf{y}}\, g(\vec{\mathbf{y}} - \vec{\mathbf{x}})\hat{a}_j^\dagger(\vec{\mathbf{y}})\hat{a}_j(\vec{\mathbf{y}}), \tag{4.1.14}$$

with $g(\vec{\mathbf{x}})$ the smearing function, a positive real spherically symmetric function peaked at $\vec{\mathbf{x}} = 0$. The smearing function must be normalized such that

$$\int \mathrm{d}^3\vec{\mathbf{x}}\, g(\vec{\mathbf{x}}) = 1, \tag{4.1.15}$$

so that

$$\int \mathrm{d}^3\vec{\mathbf{x}}\, \hat{N}_j(\vec{\mathbf{x}}) = \hat{N}_j, \tag{4.1.16}$$

where $\hat{N}_j$ is the total number operator of particle type j. Inspired by the GRW model, the smearing function is chosen to be

$$g(\vec{\mathbf{x}}) = \frac{1}{(\sqrt{2\pi}r_c)^3} e^{-\vec{\mathbf{x}}^2/2r_c^2}, \tag{4.1.17}$$

where the parameter r_c, as with the GRW model, represents the width of localization, and in this case r_c^3 is essentially the volume over which the average $\hat{N}_j(\vec{\mathbf{x}})$ is taken.

The local averaged density operator eigenspace is a Fock space with pseudo-vector elements given by the eigenvector equation $\hat{N}_j(\vec{\mathbf{x}})\,|\mathbf{q}_j\rangle = n_j(\vec{\mathbf{x}})\,|\mathbf{q}_j\rangle$ for each particle type j with

$$|\mathbf{q}_j\rangle = \mathcal{N}\hat{a}_j^\dagger(\vec{\mathbf{q}}_1)\hat{a}_j^\dagger(\vec{\mathbf{q}}_2)\cdots\hat{a}_j^\dagger(\vec{\mathbf{q}}_N)\,|0_j\rangle \quad \text{and,} \tag{4.1.18}$$

$$n_j(\vec{\mathbf{x}}) = \sum_{i=1}^{N} g(\vec{\mathbf{q}}_i - \vec{\mathbf{x}}), \tag{4.1.19}$$

with $\mathcal{N}$ a normalization constant and $\mathbf{q}_j$ a list of occupation numbers of particles of type j for each point in space.

While models of the form of (4.1.11) have been used in various works [76, 52, 50, 98], there are mostly two commonly used options in the literature for the collapse operators $\hat{A}_i$, both of which include the local averaged density operator $\hat{N}_j(\vec{\mathbf{x}})$ as the main functional component. The two options are $\hat{A}(\vec{\mathbf{x}}) = \sqrt{\gamma}\sum_j \hat{N}_j(\vec{\mathbf{x}})$, where we use the local averaged

density operator directly [74] giving a density-proportional version, or alternatively one can choose the smeared mass density operator [52, 73], $\hat{A}(\vec{\mathbf{x}}) = \frac{\sqrt{\gamma}}{m_0}\hat{M}(\vec{\mathbf{x}})$, given by

$$\hat{M}(\vec{\mathbf{x}}) = \sum_j m_j \hat{N}_j(\vec{\mathbf{x}}), \tag{4.1.20}$$

where m_j is the mass of a particle of type j, resulting in the mass-proportional version. Here we have introduced the real constants γ and m_0 which are a coupling strength and reference mass (usually taken to be equal to that of a nucleon) respectively. In both cases we associate the labels i in the collapse operators $\hat{A}_i$ with the position point in space $\vec{\mathbf{x}}$. Each of these choices then lead to the following stochastic differential equations of the form derived from equation (4.1.12),

$$\mathrm{d}\,|\psi(t)\rangle = \left[\frac{-i}{\hbar}\hat{H} - \frac{\gamma}{2}\int \mathrm{d}^3\vec{\mathbf{x}}\left(\hat{N}(\vec{\mathbf{x}}) - \left\langle \hat{N}(\vec{\mathbf{x}})\right\rangle\right)^2 \mathrm{d}t + \sqrt{\gamma}\int \mathrm{d}^3\vec{\mathbf{x}}\left(\hat{N}(\vec{\mathbf{x}}) - \left\langle \hat{N}(\vec{\mathbf{x}})\right\rangle\right)\mathrm{d}W(\vec{\mathbf{x}})\right]|\psi(t)\rangle\,, \tag{4.1.21}$$

$$\mathrm{d}\,|\psi(t)\rangle = \left[\frac{-i}{\hbar}\hat{H} - \frac{\gamma}{2m_0^2}\int \mathrm{d}^3\vec{\mathbf{x}}\left(\hat{M}(\vec{\mathbf{x}}) - \left\langle \hat{M}(\vec{\mathbf{x}})\right\rangle\right)^2 \mathrm{d}t + \frac{\sqrt{\gamma}}{m_0}\int \mathrm{d}^3\vec{\mathbf{x}}\left(\hat{M}(\vec{\mathbf{x}}) - \left\langle \hat{M}(\vec{\mathbf{x}})\right\rangle\right)\mathrm{d}W(\vec{\mathbf{x}})\right]|\psi(t)\rangle\,, \tag{4.1.22}$$

giving the density-proportional and mass-proportional version respectively. Here, as was the case in equation (4.1.12), we have used the shorthand $\langle\cdot\rangle \equiv \langle\psi|\cdot|\psi\rangle$. While both versions have the same properties in regards to their possible applications to descriptions of the measurement process (indeed, they are identical up to a particle-unique factor), the mass-proportional version attempts to relate the process of collapse to gravity [52, 73] and also results in more accurate predictions in particular scenarios where the density-proportional version fails [101]. To this end, we shall make use of this version when it is necessary to choose. Results obtained making use of this choice relating to the reduction of the state vector are similar for both versions and the conclusions apply to both.

Relating the forms of (4.1.21) and (4.1.22) back to Itô's lemma,

$$\mathrm{d}f(t,\vec{\mathbf{x}}) = \frac{\mathrm{d}f(t,\vec{\mathbf{x}})}{\mathrm{d}t}\,\mathrm{d}t + \frac{1}{2}\frac{\mathrm{d}^2 f(t,\vec{\mathbf{x}})}{\mathrm{d}x^2}\,\mathrm{d}t + \frac{\mathrm{d}f(t,\vec{\mathbf{x}})}{\mathrm{d}x}\,\mathrm{d}W(t)\,, \tag{4.1.23}$$

for some twice differential function $f(t,\vec{\mathbf{x}})$ and Wiener process $W(t)$, it can be seen from the second and third terms on the right-hand side how the change of the system wavefunction in space drives towards the expectation values of the collapse operators. Said differently, the

use of density number operators $\hat{a}_j^\dagger(\vec{\mathbf{x}})\hat{a}_j(\vec{\mathbf{x}})$ causes the suppression of higher densities $n(\vec{\mathbf{x}})$ at various points in space, which in the language of second quantization is equivalent to collapse of the system wavefunction. This can more explicitly be shown for any CSL model of the form of equation (4.1.12) as done in [22, 74] for the discrete case. Here I shall show it applied to equation (4.1.22) which is a continuous case. We assume that the rate at which reduction takes place is significantly faster than any dynamics described by the Hamiltonian, allowing us to ignore it for simplicity, giving the short time dynamics described by

$$\mathrm{d}\left|\psi(t)\right\rangle = \left[-\int \mathrm{d}^3\vec{\mathbf{x}}\left(\hat{A}(\vec{\mathbf{x}}) - \left\langle \hat{A}(\vec{\mathbf{x}})\right\rangle\right)^2 \mathrm{d}t + \int \mathrm{d}^3\vec{\mathbf{x}}\left(\hat{A}(\vec{\mathbf{x}}) - \left\langle \hat{A}(\vec{\mathbf{x}})\right\rangle\right)\mathrm{d}W(\vec{\mathbf{x}})\right]\left|\psi(t)\right\rangle, \tag{4.1.24}$$

with $\left\langle \hat{A}(\vec{\mathbf{x}})\right\rangle \equiv \left\langle\psi\right|\hat{A}(\vec{\mathbf{x}})\left|\psi\right\rangle$. For simplicity we shall work with a single particle, $\left|\mathbf{q}\right\rangle = \mathcal{N}\hat{a}^\dagger(\vec{\mathbf{q}})\left|0\right\rangle$, where we have dropped the particle type index j for the sake of ink. We then define

$$\hat{A}(\vec{\mathbf{x}}) = \frac{\sqrt{\gamma}}{m_0}\hat{M}(\vec{\mathbf{x}}) = \int \mathrm{d}^3\vec{\mathbf{q}}\, a(\vec{\mathbf{x}} - \vec{\mathbf{q}})\hat{Z}(\vec{\mathbf{q}}) \tag{4.1.25}$$

which, in the case of a single particle, is given as

$$a(\vec{\mathbf{q}} - \vec{\mathbf{x}}) = \frac{m\sqrt{\gamma}}{m_0}n(\vec{\mathbf{x}} - \vec{\mathbf{q}}) = \frac{m\sqrt{\gamma}}{m_0}g(\vec{\mathbf{x}} - \vec{\mathbf{q}}), \tag{4.1.26}$$

$$\hat{Z}(\vec{\mathbf{q}}) = \left|\mathbf{q}\right\rangle\left\langle\mathbf{q}\right|, \tag{4.1.27}$$

where m is the mass of the particle and $\vec{\mathbf{q}}$ refers to the label of the creation operator $\hat{a}^\dagger(\vec{\mathbf{q}})$, and might be interpreted as the "location" of the particle. The projection operators $\hat{Z}(\vec{\mathbf{q}})$ are orthogonal with $\int \mathrm{d}^3\vec{\mathbf{q}}\,\hat{Z}(\vec{\mathbf{q}}) = \hat{\mathbf{1}}$, where $\hat{\mathbf{1}}$ is the identity. We then consider the real non-negative values $z(\vec{\mathbf{q}})$ that describe the superposition of the system $\left|\psi\right\rangle$ in the eigenbasis $\left|\mathbf{q}\right\rangle$ of our operator $\hat{A}$,

$$\left\langle\psi\right|\hat{Z}(\vec{\mathbf{q}})\left|\psi\right\rangle = z(\vec{\mathbf{q}}), \tag{4.1.28}$$

which then has the property $\int \mathrm{d}^3\vec{\mathbf{q}}\, z(\vec{\mathbf{q}}) = 1$. Here $z(\vec{\mathbf{q}})$ could be said to describe the "spread" of the particle in space in terms of its superposition at various points $\vec{\mathbf{q}}$ in space.

Making use of these above properties, we can find that the short time dynamics of our

system described by equation (4.1.24) can then be written as

$$\mathrm{d}[\hat{Z}(\vec{\mathbf{q}})\,|\psi(t)\rangle] = \Bigg[-\int \mathrm{d}^3\vec{\mathbf{x}} \left(\int \mathrm{d}^3\vec{\mathbf{y}}\, z(\vec{\mathbf{y}}) \big(a(\vec{\mathbf{x}} - \vec{\mathbf{q}}) - a(\vec{\mathbf{x}} - \vec{\mathbf{y}}) \big) \right)^2 \mathrm{d}t + \int \mathrm{d}^3\vec{\mathbf{x}} \left(\int \mathrm{d}^3\vec{\mathbf{y}}\, z(\vec{\mathbf{y}}) \big(a(\vec{\mathbf{x}} - \vec{\mathbf{q}}) - a(\vec{\mathbf{x}} - \vec{\mathbf{y}}) \big) \right) \mathrm{d}W(\vec{\mathbf{x}}) \Bigg] \hat{Z}(\vec{\mathbf{q}})\,|\psi(t)\rangle\,. \quad (4.1.29)$$

Making use of Itô's lemma and the nature of projection operators, we also then have that

$$\mathrm{d}\,\langle\psi|\,\hat{Z}(\vec{\mathbf{q}})\,|\psi\rangle = \mathrm{d}[\langle\psi|\,\hat{Z}(\vec{\mathbf{q}})]\,\hat{Z}(\vec{\mathbf{q}})\,|\psi\rangle + \langle\psi|\,\hat{Z}(\vec{\mathbf{q}})\,\mathrm{d}[\hat{Z}(\vec{\mathbf{q}})\,|\psi\rangle] + \Big\langle \mathrm{d}[\langle\psi|\,\hat{Z}(\vec{\mathbf{q}})]\,\mathrm{d}[\hat{Z}(\vec{\mathbf{q}})\,|\psi\rangle] \Big\rangle\,, \quad (4.1.30)$$

to which we can apply the result obtained for the short time dynamics, equation (4.1.29). This gives a description of the dynamics of the superposition of $|\psi\rangle$ in the basis $|\mathbf{q}\rangle$,

$$\mathrm{d}z(\vec{\mathbf{q}}) = 2z(\vec{\mathbf{q}}) \int \mathrm{d}^3\vec{\mathbf{x}} \left(\int \mathrm{d}^3\vec{\mathbf{y}}\, z(\vec{\mathbf{y}}) \big(a(\vec{\mathbf{x}} - \vec{\mathbf{q}}) - a(\vec{\mathbf{x}} - \vec{\mathbf{y}}) \big) \right) \mathrm{d}W(\vec{\mathbf{x}})\,. \quad (4.1.31)$$

This gives a qualitative sense for the diffusion of $z(\vec{\mathbf{q}})$ in space. We can better investigate this diffusion by using equation (4.1.31) above to find that

$$\mathrm{d}z^2(\vec{\mathbf{q}}) = 2z(\vec{\mathbf{q}})\mathrm{d}z(\vec{\mathbf{q}}) + \langle \mathrm{d}z(\vec{\mathbf{q}})\mathrm{d}z(\vec{\mathbf{q}}) \rangle \quad (4.1.32)$$

$$= 2z(\vec{\mathbf{q}})\mathrm{d}z(\vec{\mathbf{q}}) + \int \mathrm{d}^3\vec{\mathbf{x}} \left(2z(\vec{\mathbf{q}}) \int \mathrm{d}^3\vec{\mathbf{y}}\, z(\vec{\mathbf{y}}) \big(a(\vec{\mathbf{x}} - \vec{\mathbf{q}}) - a(\vec{\mathbf{x}} - \vec{\mathbf{y}}) \big) \right)^2 \mathrm{d}t\,, \quad (4.1.33)$$

where the first line is Itô's lemma. This then allows us to study the dynamics of the variance,

$$\mathrm{d}\left\langle z^2(\vec{\mathbf{q}}) \right\rangle = \left\langle \mathrm{d}z^2(\vec{\mathbf{q}}) \right\rangle \quad (4.1.34)$$

$$= \int \mathrm{d}^3\vec{\mathbf{x}} \left(2z(\vec{\mathbf{q}}) \int \mathrm{d}^3\vec{\mathbf{y}}\, z(\vec{\mathbf{y}}) \big(a(\vec{\mathbf{x}} - \vec{\mathbf{q}}) - a(\vec{\mathbf{x}} - \vec{\mathbf{y}}) \big) \right)^2 \mathrm{d}t\,, \quad (4.1.35)$$

where in the first line the short time assumption, $\hat{H} = 0$, has been used. From this result we find the variance to be strictly non-negative, $\frac{\mathrm{d}}{\mathrm{d}t}\langle z^2(\vec{\mathbf{q}}) \rangle \geq 0$. Taking into account the boundedness property of $\langle z^2(\vec{\mathbf{q}}) \rangle \leq 1$, this immediately implies that after some sufficiently long time

$$\int \mathrm{d}^3\vec{\mathbf{y}}\, z(\vec{\mathbf{y}}) \big(a(\vec{\mathbf{x}} - \vec{\mathbf{q}}) - a(\vec{\mathbf{x}} - \vec{\mathbf{y}}) \big) \xrightarrow[t\to\infty]{} 0 \qquad \forall \vec{\mathbf{x}}, \vec{\mathbf{q}}. \quad (4.1.36)$$

We can now find the $z(\vec{\mathbf{y}})$ that solves this equation at long times when equality to 0 is reached. To do so we Fourier transform equation (4.1.36) above into an algebraic equation, which we can then solve and perform an inverse Fourier transform, leading to

$$z(\vec{\mathbf{y}}) \xrightarrow[t\to\infty]{} \delta^3(\vec{\mathbf{y}} - \vec{\mathbf{q}}). \quad (4.1.37)$$

This then gives directly from equation (4.1.27) and (4.1.28) that

$$\|\langle\vec{\mathbf{x}}|\psi\rangle\| \xrightarrow[t\to\infty]{} \delta^3(\vec{\mathbf{x}}-\vec{\mathbf{q}}), \tag{4.1.38}$$

showing the act of collapse as the dynamical reduction of the state vector $|\psi\rangle$ to a single basis state $|\vec{\mathbf{q}}\rangle$ of the collapse operator $\hat{A}$. This is a description of the collapse in a manner that is postulated in standard Quantum mechanics and "hard-coded" in the GRW model. Instead, we see this behaviour as emergent in the CSL model.

We can apply this result to the expectation value of our observable $\hat{A}(\vec{\mathbf{x}}) = \frac{\sqrt{\gamma}}{m_0}\hat{M}(\vec{\mathbf{x}})$, to find that the mass of the particle is distributed according to

$$\left\langle \hat{A}(\vec{\mathbf{x}}) \right\rangle \to a(\vec{\mathbf{x}}-\vec{\mathbf{q}}), \tag{4.1.39}$$

$$\left\langle \hat{M}(\vec{\mathbf{x}}) \right\rangle \to \frac{m}{(\sqrt{2\pi}r_c)^3} e^{-(\vec{\mathbf{x}}-\vec{\mathbf{q}})^2/2r_c^2}, \tag{4.1.40}$$

showing the eventual reduction of the mass distribution of the particle to a packet of mass centred on some point $\vec{\mathbf{q}}$.

The probability of each point being chosen can also be found. Noting the martingale property $\mathrm{d}\langle z(\vec{\mathbf{q}})\rangle = \langle \mathrm{d}z(\vec{\mathbf{q}})\rangle = 0$ of $z(\vec{\mathbf{q}})$ from equation (4.1.31), we also have that

$$\langle z(\vec{\mathbf{q}})\rangle = z(\vec{\mathbf{q}}, t=0) = \|\langle \mathbf{q}|\psi(t=0)\rangle\|^2, \tag{4.1.41}$$

which is exactly the probability one would have expected from Born's rule. Here we see that the CSL model offers a means of state vector reduction that reproduces the same results as standard quantum mechanics about measurement outcomes without any *ad hoc* addition of collapse postulates. This is not merely a suppression of correlations between states, this demonstrates a suppression of superposition leading to a single outcome for a single particle.

In practice, for the case of a single particle, this would take an enormous amount of time and the short time approximation $\hat{H} = 0$ would not be valid. Instead, a balance is found between the effects of the Hamiltonian and the stochastic terms with the particle wavefunction reaching some constant spread [26, 74, 41]. This is discussed in the context of the QMUPL model in the next section.

We can investigate the rate with which this localization process takes place on average via the use of the systems density matrix $\rho = |\psi\rangle\langle\psi|$. The off-diagonal terms of this density

matrix in the position basis, $\langle \mathbf{q}' | \rho | \mathbf{q}'' \rangle$, will be suppressed by the reduction process, giving us a sense of the rate at which the mechanism works.

Similarly to what was shown in the case of the GRW model in equations (4.1.8) and (4.1.9), the redefining of the system in the CSL model into relative and centre of mass basis shows that that the reduction occurs only to the centre of mass basis [74]. In a similar fashion to GRW we will see that the rate of reduction scales with the size of the system, describing an amplification mechanism for reduction. The nature of the scaling, however, is not the same, though still fast enough that for the short time approximation, $\hat{H} = 0$, to still be valid.

We find that the decay of the off-diagonal terms of the density matrix ρ of a many-particle system in the position basis is then given by

$$\frac{\partial}{\partial t} \langle \mathbf{q}' | \rho | \mathbf{q}'' \rangle = -\Gamma(\mathbf{q}', \mathbf{q}'') \langle \mathbf{q}' | \rho | \mathbf{q}'' \rangle , \tag{4.1.42}$$

where we assume a system of entirely a single type of particle j with mass m, giving $|\mathbf{q}'\rangle = \mathcal{N} \hat{a}_j^\dagger(\vec{\mathbf{q}}_1') \hat{a}_j^\dagger(\vec{\mathbf{q}}_2') \cdots \hat{a}_j^\dagger(\vec{\mathbf{q}}_N') |0\rangle$, where $\mathbf{q}' = \vec{\mathbf{q}}_1', \vec{\mathbf{q}}_2', \cdots, \vec{\mathbf{q}}_N'$ (and similarly for $\mathbf{q}''$). The decay function Γ is given by

$$\Gamma(\mathbf{q}', \mathbf{q}'') = \frac{\gamma}{2} \left(\frac{m}{m_0} \right)^2 \sum_{i,j}^{N} \left[G(\vec{\mathbf{q}}_i' - \vec{\mathbf{q}}_j') + G(\vec{\mathbf{q}}_i'' - \vec{\mathbf{q}}_j'') - 2G(\vec{\mathbf{q}}_i' - \vec{\mathbf{q}}_j'') \right] , \tag{4.1.43}$$

with i and j the indices of each particle in the system, and

$$G(\vec{\mathbf{y}}_i - \vec{\mathbf{y}}_j) = \int \mathrm{d}^3 \vec{\mathbf{x}} \, g(\vec{\mathbf{y}}_i - \vec{\mathbf{x}}) g(\vec{\mathbf{y}}_j - \vec{\mathbf{x}}), \tag{4.1.44}$$

$$= \frac{1}{(4\pi r_c^2)^{3/2}} e^{-(\vec{\mathbf{y}}_i - \vec{\mathbf{y}}_j)^2 / 4 r_c^2}. \tag{4.1.45}$$

In the case of a system consisting of a single particle, the decay function Γ becomes

$$\Gamma(\mathbf{q}', \mathbf{q}'') = \frac{\gamma}{(4\pi r_c^2)^{3/2}} \left(\frac{m}{m_0} \right)^2 \left[1 - e^{-(\vec{\mathbf{q}}' - \vec{\mathbf{q}}'')^2 / 4 r_c^2} \right] , \tag{4.1.46}$$

which is exactly what was found for the single particle collapse of the GRW model, equation (4.1.10), giving us a rate of collapse in terms of γ as

$$\lambda_{CSL} = \frac{\gamma}{(4\pi r_c^2)^{3/2}} \left(\frac{m}{m_0} \right)^2 . \tag{4.1.47}$$

For a system of very few particles, the collapse given by the CSL model is similar to that of the GRW model. In the case of a more complex system containing many particles, these

results no longer align quite as well. In the case of particles in clusters such that distances are small, $|\mathbf{q}' - \mathbf{q}''| \ll r_c$, we find via Taylor expansion of $G(\vec{\mathbf{x}})$,

$$G(\vec{\mathbf{x}}) \approx \frac{1}{(4\pi r_c^2)^{3/2}} \left[1 - \frac{\vec{\mathbf{x}}}{4r_c^2}\right], \tag{4.1.48}$$

leading to a decay function Γ of

$$\Gamma(\mathbf{q}', \mathbf{q}'') \approx \frac{\lambda_{CSL}}{4r_c^2} \left[\sum_i^N (\vec{\mathbf{q}}_i' - \vec{\mathbf{q}}_i'')\right]^2. \tag{4.1.49}$$

We find a quadratic dependence on the number of particles within a cluster n. We also find a quadratic dependence on the distance between points of superposition, as in the GRW model. From equation (4.1.46) applied to the centre of mass of each such cluster that are a distance apart, $|\mathbf{q}' - \mathbf{q}''| \geq r_c$, we see that $\Gamma \sim \lambda_{CSL}$ per cluster. For N such clusters, we see that the rate of localization is linear in N in a manner similar to the GRW model. As such we can approximate the rate of collapse of N clusters of n particles as [2]

$$\Gamma = \lambda_{CSL} n^2 N. \tag{4.1.50}$$

Here we see the amplification mechanism in action. An increased rate of collapse for larger systems. It is worth noting that the quadratic dependence on the number of particles n in the clusters is a result of the particles being identical. This effect is absent in the GRW model.

Another formulation for the collapse rate of rigid macroscopic bodies is obtained if we make use of a density function $D(\vec{\mathbf{x}})$ as an expression for the mass distribution [74]. Here the density function averages over the number of particles per unit space. In the case where the macroscopic object is a homogeneous structure with constant density D, the decay function becomes

$$\Gamma(\vec{\mathbf{X}}, \vec{\mathbf{X}}') = \gamma D n_{out}, \tag{4.1.51}$$

with the object's centre of mass being in a superposition of $\vec{\mathbf{X}}$ and $\vec{\mathbf{X}}'$. n_{out} is then the number of particles that are within the volume of the object when it is positioned at $\vec{\mathbf{X}}$ but outside the volume of the object if it were positioned at $\vec{\mathbf{X}}'$.

The CSL model is currently the most advanced collapse model to date. Its properties have been further explored in other works [26, 22, 74].

4.1.3 QMUPL model

For all the strengths of the CSL model, it has some issues in its use since it can be incredibly difficult to find analytical solutions to it. The so-called quantum mechanics with universal position localization (QMUPL) model is the middle ground between mathematical simplicity and physical realism. It was first introduced as a modification of the GRW model by Diósi [52, 53] where the same axioms as those of the GRW model were used, but the dynamics are instead described by a stochastic differential equation of the form

$$\mathrm{d}\left|\psi(t)\right\rangle = \left[-\frac{i}{\hbar}H\mathrm{d}t + \sum_{i=1}^{N}\sqrt{\lambda_i}(\hat{q}_i - \langle\hat{q}_i\rangle)\mathrm{d}W_i(t) - \frac{1}{2}\sum_{i=1}^{N}\lambda_i(\hat{q}_i - \langle\hat{q}_i\rangle)^2\mathrm{d}t\right]\left|\psi(t)\right\rangle, \quad (4.1.52)$$

where H is the standard quantum Hamiltonian of the system $\psi(t)$ in some Hilbert space $\mathcal{H}_S$ of N distinguishable particles. The $\hat{q}$ are the position operators of the i-th particle in the system, and $W_i(t)$ are independent Wiener processes for each particle in the system. Here $\langle\cdot\rangle$ is the quantum expectation value. The collapse constant λ_i is chosen to be proportional to the mass m_i of its associated particle,

$$\lambda_i = \frac{m_i}{m_0}\lambda_0, \quad (4.1.53)$$

where m_0 is a reference mass, usually taken to be the nucleon's mass and λ_0 is the collapse strength.

It has been assumed here that the evolution of the system is norm preserving and that superluminal propagation is impossible. These assumptions act to restrict the relationship between the coefficient of the stochastic term and that of the drift term resulting in the coefficients as written above [26].

The lack of a second free parameter (r_c in the CSL and GRW model) comes from the lack of spacial dependence of the stochastic fluctuations which only fluctuate in time. Regardless, the stochastic process acts continuously on the system. This is a property that is shared with the CSL model, but not the GRW where the stochastic jumps are sudden and discontinuous in nature. This makes the QMUPL model a middle ground between these two models, with it being shown to be a scaling limit of the GRW process [58], and its similarity to the form of the CSL model being rather striking. The relationship of the QMUPL model with the GRW

and CSL model allows us to make use of the relative mathematical simplicity of the QMUPL model to offer suggestions for the properties of the more mathematically cumbersome GRW and CSL models. This use of the QMUPL as a toy model makes it a useful tool in the investigation of properties of spontaneous collapse models. We shall use this to show the spreading of the wavefunction that was mentioned in the CSL model above but within the context of the QMUPL model. A similar discussion was had by Collett and Pearle [41] in the context of the CSL model.

Consider the case of a system consisting of a single free particle given by the Gaussian state

$$\langle \vec{\mathbf{q}}|\psi(t)\rangle = \mathcal{N}\exp\Big\{-a(t)\big(\vec{\mathbf{q}}-\vec{\mathbf{x}}(t)\big)^2 + i\vec{\mathbf{k}}(t)\vec{\mathbf{q}}\Big\}, \tag{4.1.54}$$

where $a(t)$ is a complex function that encodes the spread of position and momentum in time, $\mathcal{N}$ a normalization constant, with $\vec{\mathbf{x}}$ and $\vec{\mathbf{k}}$ being taken as real valued functions of time. By inserting this into equation (4.1.52), one can show it to be a solution given that a system of stochastic differential equations for the time dependent functions $a(t)$, $\vec{\mathbf{x}}(t)$, and $\vec{\mathbf{k}}(t)$ hold true [14]. In the case of a Gaussian as we have here, we also have that $\vec{\mathbf{x}}(t) = \langle\vec{\mathbf{q}}\rangle_t$ is the mean position of the packet at time t, and $\vec{\mathbf{k}}(t) = \langle\vec{\mathbf{p}}\rangle_t/\hbar$ is the mean wave-number of the wavefunction. This then leads to the time dependent functions $a(t)$, $\langle\vec{\mathbf{q}}\rangle_t$, and $\langle\vec{\mathbf{p}}\rangle_t$ being solutions to the stochastic differential equations

$$\mathrm{d}a(t) = \left[\lambda - \frac{2i\hbar}{m}a^2(t)\right]\mathrm{d}t\,, \tag{4.1.55}$$

$$\mathrm{d}\langle\vec{\mathbf{q}}\rangle_t = \frac{\langle\vec{\mathbf{p}}\rangle_t}{m}\,\mathrm{d}t + \sqrt{\lambda}\frac{1}{2a_{\mathbb{R}}(t)}\,\mathrm{d}W(t)\,, \tag{4.1.56}$$

$$\mathrm{d}\langle\vec{\mathbf{p}}\rangle_t = -\hbar\sqrt{\lambda}\frac{a_{\mathbb{I}}(t)}{a_{\mathbb{R}}(t)}\,\mathrm{d}W(t)\,, \tag{4.1.57}$$

with $a_{\mathbb{R}}(t)$ and $a_{\mathbb{I}}(t)$ being the real and complex component of $a(t)$ respectively, $a(t) = a_{\mathbb{R}}(t) + ia_{\mathbb{I}}(t)$. From this we can make use of the deterministic nature of equation (4.1.55) to find an expression for the deviation of the position and momentum of the Gaussian particle,

$$\sigma^2_{\vec{\mathbf{q}}}(t) = \langle\vec{\mathbf{q}}^2\rangle - \langle\vec{\mathbf{q}}\rangle^2 = \frac{1}{4a_{\mathbb{R}}(t)} = \frac{\hbar}{m\omega}\frac{\cosh(\omega t+\theta_1)+\cos(\omega t+\theta_2)}{\sinh(\omega t+\theta_1)+\sin(\omega t+\theta_2)}, \tag{4.1.58}$$

$$\sigma^2_{\vec{\mathbf{p}}}(t) = \langle\vec{\mathbf{p}}^2\rangle - \langle\vec{\mathbf{p}}\rangle^2 = \hbar^2\frac{\|a(t)\|^2}{a_{\mathbb{R}}(t)} = \frac{\hbar m\omega}{2}\frac{\cosh(\omega t+\theta_1)-\cos(\omega t+\theta_2)}{\sinh(\omega t+\theta_1)+\sin(\omega t+\theta_2)}, \tag{4.1.59}$$

where $\omega = 2\sqrt{\hbar\lambda_0/m_0}$, with the parameters θ_1 and θ_2 dependent on the initial conditions. Here we note that setting $\lambda_0 = 0$ recovers the results from the standard Schrödinger equation.

These descriptions of the spread of position and momentum in time, taken to their long time limit, $t \to \infty$, then show the asymptotic behaviour of these quantities. Here we see that they do not increase indefinitely, as one might expect from a pure Schrödinger description, but reach a final asymptotic value,

$$\sigma^2_{\vec{\mathbf{q}}}(t \to \infty) = \frac{\hbar}{m\omega}, \tag{4.1.60}$$

$$\sigma^2_{\vec{\mathbf{p}}}(t \to \infty) = \frac{\hbar m\omega}{2}. \tag{4.1.61}$$

For a $\lambda_0 \approx 10^{-2}$ m^{-2}s^{-1}, which is the value suggested by Bassi [14] to reproduce the same results as the GRW model, these quantities become $\sigma^2_{\vec{\mathbf{q}}}(t \to \infty) \approx m^{-1} \times 10^{-30}$ and $\sigma^2_{\vec{\mathbf{p}}}(t \to \infty) \approx m \times 10^{-38}$, where m is the mass of the particle in kg. This then demonstrates how macroscopic objects can have nearly point-like position and momentum. The spreads of position and momentum stabilize at a value that compromises between the continuous spreading of the Schrödinger equation and the shrinking due to the dynamical collapse caused by the Stochastic modification to the dynamics. Further, this compromise settles on a value that is nearly the minimum allowed by Heisenberg's uncertainty relations,

$$\sigma_{\vec{\mathbf{q}}}(\infty)\sigma_{\vec{\mathbf{p}}}(\infty) = \frac{\hbar}{\sqrt{2}}. \tag{4.1.62}$$

As such, we see that as a by-product of collapse in the position basis, one also has that the system undergoes a dynamical collapse in the momentum basis, subject to Heisenberg's uncertainty relations.

While the differential equation for $a(t)$ is deterministic, only indirectly being affected by the stochastic properties of the dynamics via the inclusion of λ, the differential equations for $\langle\vec{\mathbf{q}}\rangle$ and $\langle\vec{\mathbf{p}}\rangle$ include stochastic diffusion terms. These properties undergo a Brownian motion due to the collapse noise "kicking" the wavefunction about. These "kicks" scale inversely to the size of the object, with the diffusion becoming negligible for larger objects. The averages of these values then follow the *classical* equations for a free particle,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\langle\vec{\mathbf{q}}\rangle_t\right\rangle = \frac{\left\langle\langle\vec{\mathbf{p}}\rangle_t\right\rangle}{m}, \tag{4.1.63}$$

$$\left\langle\langle\vec{\mathbf{p}}\rangle_t\right\rangle = \langle\vec{\mathbf{p}}\rangle_0\,. \tag{4.1.64}$$

This demonstrates how stochastic collapse models can describe the dynamics of microscopic systems according to the standard Schrödinger equation, as well as the dynamics of macroscopic systems according to the Newton's equations of motion.

We can also investigate the energy of the system via the stochastic differential for the expectation value of the Hamiltonian, $\langle H \rangle_t \equiv \langle \vec{\mathbf{p}} \rangle_t / 2m$,

$$\mathrm{d} \langle H \rangle_t = \frac{\lambda \hbar^2}{2m} \, \mathrm{d}t - \sqrt{\lambda} \frac{\hbar}{m} \frac{a_{\mathbb{I}}(t)}{a_{\mathbb{R}}(t)} \langle \vec{\mathbf{p}} \rangle_t \, \mathrm{d}W(t) \,, \tag{4.1.65}$$

from which we can see a change in the energy with time,

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \langle H \rangle_t \rangle = \frac{\lambda_0 \hbar^2}{2m_0} \approx 10^{-43} \ \mathrm{Js}^{-1}. \tag{4.1.66}$$

This lack of energy conservation is a common feature amongst space-collapse models, with whatever is causing the Brownian motion of the dynamics of the particle imparting small amounts of energy into it. However, the amount of energy, given the value for λ_0 given above, is currently too small for any existing technology to observe.

So far this investigation has been focused on a system described by a Gaussian wavefunction, equation (4.1.54). This would idealize the above description as a special case. However, it was found that the QMUPL model acts on any initial state to smooth it out over time into the form of a Gaussian wavefunction described here [18, 17]. As such we see that for any initial system, the wavefunction evolves into a Gaussian function with deviations described by equations (4.1.58) and equation (4.1.59) due to constant "kicks" from a stochastic noise term. This same stochastic effect causes a reduction of the wavefunction that eventually balances with the spreading due to the standard Schrödinger equations, resulting in these spreads reaching an asymptotic constant value.

The nature of the QMUPL model as a scaling limit of the GRW model, as well as the CSL model tailored to reproduce the results of the GRW model, suggest that these conclusions should also hold for the two other models discussed here.

4.2 Generalizations

The models of dynamical state vector reduction so far investigated have made use of the addition of nonlinear stochastic modifications to the Schrödinger equation. It has been

shown that the models successfully describe state vector reduction with the appropriate probabilities. Additionally, these models successfully recreate the dynamics of micro- and macroscopic systems up to our ability to resolve said dynamics.

These models offer consistent solutions to the measurement problem up to our technological ability to measure any disparity between them and standard quantum mechanics. This future ability to possibly distinguish the differences between the predictions of standard quantum mechanics and the results predicted by the collapse models offer hope for the possibility to resolve the validity of these models. Until then there is still a fair amount of space within the resolution of our current technologies for various modifications to be made to the dynamical collapse models while still offering valid outcomes that match experimental results. This is all a direct result of these models not being based on any underlying theories, without which there is no theoretical reason for any particular choices made in the model other than their ability to make accurate predictions.

4.2.1 The Parameters

The most obvious place where the lack of experimental resolution is best noted is in the broad range of parameters λ and r_c predicted for each model. The values of these parameters must only ensure that the results of the model agree with any observations about the microscopic behaviour of standard quantum mechanics while also ensuring the localization of macroscopic objects such that their dynamics can be accurately modelled by Newton's equations of motion. As such, the choice of parameter very much comes down to the nature of the system under investigations and the experimenters choice of what rate of collapse for some number of particles seems reasonable.

In determining the numerical values of the parameters λ and r_c, we note that physically significant effects of the dynamics of macroscopic systems that result from the stochastic modifications introduced to the Schrödinger equation in each model are rather dependent on the value of Λ, which we introduce here for the sake of comparison, where for the dynamics

of a single particle we have [22, 75, 14]

$$\Lambda_{GRW} = \frac{\lambda_{GRW}}{r_c^2}, \tag{4.2.1}$$

$$\Lambda_{CSL} = \frac{\gamma}{(4\pi)^{3/2} r_c^5} = \frac{\lambda_{CSL}}{r_c^2}, \tag{4.2.2}$$

$$\Lambda_{UPL} = 2\lambda_{UPL}, \tag{4.2.3}$$

with UPL being a shorthand for QMUPL. In the case where the particle is well localized below r_c, by which we mean that $\langle \vec{\mathbf{x}} | \, \rho \, | \vec{\mathbf{y}} \rangle \gg 0$ only for values $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ such that $|\vec{\mathbf{x}} - \vec{\mathbf{y}}| \ll r_c$, the values of Λ for each model happen to coincide, $\Lambda_{GRW} = \Lambda_{CSL} = \Lambda_{UPL}$ [16]. It is also interesting that the r_c^5 appearing in the denominator of the CSL case is of the order of the plank length $r_c^5 \sim l_p \approx 10^{-35}$ m.

The parameters λ and r_c, as has been mentioned already, are the average rate of localization and localization distance respectively. In choosing these parameters, there are several criteria to keep in mind. In choosing the localization distance r_c, we wish to keep in mind that should an atom in a larger system undergo localization, it should not in any way affect the internal structure of the system in question and that the decoupling of the centre of mass and relative coordinates are still valid. As such, we wish for r_c to be large with respect to inter-atomic dimensions ($\sim 10^{-10\pm1}$m). Comparatively, we wish r_c to be of an order that we cannot have macroscopic objects localizing to a superposition. These considerations lead Ghirardi, Rimini, and Weber [75] to choose for r_c the quantity

$$r_c \simeq 10^{-7} \text{ m}, \tag{4.2.4}$$

which while honestly rather arbitrary has been used consistently by authors since. Adler [2], in his attempt to bound these parameters in the context of the CSL model, has pointed out that based on latent image formation in the eye, an argument could be made for the value of r_c to be bound from above at $r_c \sim< 10^{-8}$m. Some other less successful models not included here have also found this choice for r_c via various means [51, 29]. For whatever the choice of r_c, the value of λ can be chosen to compensate in order to still have predictions match experimental results.

As for finding some quantity for the average rate of collapse, λ, we can study the rate of decay of the off-diagonal terms of the density matrix Γ. This is given for the CSL model in equa-

tions (4.1.50) and (4.1.51) for two different systems and the GRW model by $\Gamma_{GRW} = N\lambda_{GRW}$, for a system of N particles. As for Γ_{UPL} for the QMUPL model, it is studied in the double Gaussian solution to equation (4.1.52) by Bassi in [16] and repeated in [26]. Regardless, by matching this decay with experimental results, one can determine possible values for λ. Due to the differences in each model, this value is model dependent in contrast to r_c. Γ must be found such that for large N the decay is fast enough to ensure the localization of macroscopic systems while keeping λ itself as small as possible to ensure minimal modifications to microscopic dynamics. Additionally, we wish to minimize Λ as to reduce the effects of the stochastic modifications on the dynamics of macroscopic system. Ghirardi, Rimini, and Weber [75] chose for $\lambda_{GRW} \simeq 10^{-16}\ \mathrm{s}^{-1}$ which corresponds to $\Gamma_{GRW} \simeq 10^{7}\ \mathrm{s}^{-1}$ for a system with a number of particles of the order of Avogadro's number $N \simeq 10^{23}$. Once more, this value is chosen nearly arbitrarily, however its similarity to the age of the universe ($\sim 1.38 \times 10^{16}$ years) is suggestive. Following these choices by Ghirardi, Rimini, and Weber, the choices for parameters of following models have aimed to recreate the predictions of GRW and as such have chosen there values to match these. As such we have for each of the models

$$\lambda_{GRW} \simeq 10^{-16}\ \mathrm{s}^{-1}, \tag{4.2.5}$$

$$\lambda_{CSL} \simeq 10^{-17}\ \mathrm{s}^{-1}, \tag{4.2.6}$$

$$\lambda_{UPL} \simeq 10^{-2}\ \mathrm{m}^{-2}\mathrm{s}^{-1}. \tag{4.2.7}$$

Here $\lambda_{UPL} = \lambda_{GRW}/r_c^2$. While these values are able to reproduce the results predicted by standard quantum mechanics up to our ability to measure said results, as well as reproduce classical dynamics up to any reasonable time scale, there is still a fair bit of space for these results to change, in fact, it has been noted that an increase of λ_{CSL} of up to 10^{12} will still reproduce experimental results [2]. Only by somehow narrowing the experimental gap between macroscopic and microscopic can these values begin to be refined.

The choice of parameters by GRW are based on the assumption that the localization process is tied to the amplification process. Localization rate becomes faster for increasing system sizes. Adler, based on a discussion by Gisin and Percival [78], instead assumes that localization during detection of the particle happens prior to any amplification mechanism [2]. This different assumption is studied in the context of the mass-proportional CSL model

applied to latent image formation on either a photographic plate or solid-state track detector. Under this assumption, it is shown that the reduction rate is a factor of order $10^{9\pm2}$ too slow to account for the latent image formation.

4.2.2 Non-white noise

The stochastic modifications made in the models we have investigated depend on an underlying noise to drive the localization of the system state vector. It has been shown that in the case where this noise is chosen to be white noise the predictions that follow from these models have been able to recreate results obtained from experiments.

The choice of white noise here was made mostly due to the comparative mathematical simplicity it allows. However, there are three main reasons that one might wish to generalize the nature of the noise: Firstly, one might wish to investigate the role the choice of noise plays in the reduction of the state vector and related properties of the model [22, 19, 20, 6, 7, 23, 8]. Secondly, there are not many cases of white noise appearing in nature with the choice being made mostly for ease of calculation. In attempting to associate the noise with a physical field, the noise field could be assigned some temperature and spectrum, perhaps with cosmological properties [7, 15, 26]. Alternatively, attempts have been made to associate the noise with a gravitational source [52, 73, 70] with varying degrees of success. A third reason for generalizing the noise field, and related to the first, is one might wish to investigate how the noise affects the evolution of physical properties of the system such as energy. This question is of particular importance to the generalizing of the collapse models to the relativistic case where replacing the white noise with more general choices is a possible solution to the infinite increase in energy that results in those models [72, 22, 26].

In general, these changes to the nature of the noise still retain many of the nice features of the white noise case. State vector reduction still occurs for any matrix F, where

$$F_{ij} = \lim_{t\to\infty} \int_0^t \mathrm{d}s\, D_{ij}(t,s), \tag{4.2.8}$$

such that F is positive definite, with $D_{ij}(t_1,t_2) = \langle\langle W_i(t_1)W_j(t_2)\rangle\rangle$ being the correlation function of the stochastic process $W(t)$ driving the localization of the state vector [6]. The rate at which this reduction takes place is then dependant on the nature of the correlation

function [7]. Other features of the models with white noise such as the dynamics of the system's physical properties are also shown to be recreated [22, 20, 19]. This includes the increase in energy over time that is characteristic of the white noise models [19, 6], unless the chosen noise correlation function has a low frequency cutoff as is associated with thermal noise [7, 26].

A remaining shortfall in each of these possible sources is that none of them are yet to account for the non-Hermitian nature of the coupling. There is still work to be done without any conclusion as to whether generalizing the noise in these directions indeed offers solutions to the problems raised above.

4.2.3 Dissipative modifications

A resurfacing problematic feature of the models presented is the constant increase in energy of the system due to the infinite temperature of the white noise governing the reduction mechanism. In the non-relativistic case this increase in energy is minuscule and too small to measure with current technologies. Regardless, this is a rather undesirable property, with the energy of the system diverging in the long time limit. In an attempt to reign in this energy increase, a further dissipative modification can be made to the models. This is commonly done via a coupling to the system's momentum, which in the simpler QMUPL model's case takes the form of replacing the position operator $\hat{\vec{\mathbf{q}}}$ with

$$\hat{\vec{\mathbf{q}}} \to \hat{\vec{\mathbf{q}}} + i\frac{\mu}{\hbar}\hat{\vec{\mathbf{p}}}, \tag{4.2.9}$$

where μ is the strength of the dissipation effect [65, 66, 25, 24]. Similar modifications have been made to the CSL [115] and GRW model [116]. The additional dissipative terms are shown to still result in the reduction of the state vector, with the models still accurately modelling both macro- and microscopic dynamics, up to our ability to measure them. Additionally, the limiting of the spread of physical quantities are still found, with the asymptotic limit being slightly altered and reached slower [65, 66]. The modification still recreates all the features of the standard model that we want, except it acts to circumvent the increase in energy that was so problematic.

Introduction of the dissipative modification to the stochastic collapse models results in

a master equation with the same form as that of quantum Brownian motion with friction and diffusion effects included, and positively of the statistical operator at all times [64, 24, 25]. This same master equation also appears in the collisional model for environmental decoherence. From this master equation, one finds that the time evolution of the energy for a single free particle is given by

$$\left\langle\left\langle \hat{H}(t) \right\rangle\right\rangle = \left(\left\langle\left\langle \hat{H}(0) \right\rangle\right\rangle - E\right) e^{-\xi t} + E, \tag{4.2.10}$$

where ξ is the energy relaxation rate and E is the asymptotic energy of the system. Both values depend on the model in question [115, 115, 25], but continuing with the example of the QMUPL model above, we have $\xi = 4\lambda\mu$ and $E = \hbar^2/8m\mu$ with m the mass of the system [25, 24]. As one can see, the dissipative modifications in no way resulted in the case of energy conservation. Instead, the energy no longer diverges, rather reaching an asymptotic value E. This can be naturally interpreted as the stochastic process acting to thermalize the system to a temperature T, which for the QMUPL model is given by

$$T = \frac{\hbar^2}{4m\mu k_B}, \tag{4.2.11}$$

with k_B the Boltzmann constant. Using the comparison of this system's master equation to that of quantum Brownian motion, in which the friction similarly acts to thermalize the system, one can associate this temperature to that of the noise field [64, 24]. As for the choice of the temperature, it has been shown that reduction occurs for temperatures of the order $T \approx 10^{-1}$K, allowing for the possibility of very cold noise fields comparable to cosmological sources ($T \approx 2.27$K) such as the CMB and other relic background fields [18].

4.3 The Tail Problem

For all the success of collapse models, the process of localization still fails to properly reduce the system, opting for a minimum uncertainty final state instead that is still not perfectly localized in space. The final state after the process of reduction described by collapse models is not an eigenstate of the position operator. This is perfectly acceptable if one weakens the assumed relationship between the measured eigenvalue and eigenstate of the system. This

would allow for some small degree of uncertainty in the final state of the system to still describe a physical quantity, weakening the description of objectivity described in section 2.3.2.

However, even with a change in the idea of objectivity that would be required, the lack of true orthogonality of the final states, regardless of how close to true orthogonality it might be, still results in the breakdown of arithmetic for large numbers of macroscopic systems as pointed out by Lewis [87]. Briefly, if the final state of the system $|s\rangle$ is approximately in the state $|1\rangle$, given by $\langle s|1\rangle = a$ where $a \approx 1$, then for n such states, the combined state of all these systems $|s\rangle_n$ being described by the combined state $|1\rangle_n$ where all the individual states are in $|1\rangle$ is then given by ${}_n\langle s|1\rangle_n = a^n$ which for $0 \ll a < 1$ implies that ${}_n\langle s|1\rangle_n \to 0$ for n large enough.

This is what is referred to as the tails problem in collapse models. It is usually described via the lack of perfect reduction of the systems state resulting in interference terms in the density matrices that are not completely suppressed for finite times, hence the state of the system at these times having a non-zero "tail". Non-zero interference in such a case would imply a reduced, perhaps macroscopic system that exists in a superposition and could, hypothetically, be measured as such. The probability of such a thing occurring is usually impossibly small however. This problem is also present in the Environmental Decoherence scheme as the orthogonality of environmental states is only reached at the infinite time limit.

The difference arises due to how the final density matrices of each are interpreted. Collapse models result in a density matrix that is a classically statistical mixture of pure states, referred to as a proper mixed state. In the case of environmental decoherence, the final density matrix cannot be interpreted this way as it results from a partial trace. The density matrix then is referred to as an improper mixed state due to it still containing quantum correlations and the system still existing in a quantum superposition, however, the interference is instead hidden from any measurement that might be made on the local system. This distinction is an important one.

This difference in the interpretations of the final states makes this problem one that disproportionately affects collapse models since they describe an actual reduction mechanism as compared to an apparent one. The final state cannot as easily be referred to as a proper

mixture when quantum correlations still exist between the pure states.

5. Discussion

We have up to this point investigated the nature of the measurement problem, and discussed environmental decoherence and the Dynamical Reduction formalism. Individually, these chapters, while related through their application to the quantum to classical transition, are disjoint in their content. Each one could be read individually without much bearing on the other. Here we intend to speculate as to the interplay of these three subjects, with an emphasis on this speculatory nature. The intention of this thesis was a review of these above materials, as such these ideas will not be pursued here.

5.1 Collapse models, Decoherence, and the Problem of Measurement

Stating the merits of collapse models and the environmental decoherence scheme in regards to resolving aspects of the measurement problem, it is noted that each has something in which their resolution is not as satisfactory. For the collapse model program, it is its approach to the problem of Basis Ambiguity as described in section 2.3.1. While the program does indeed fully resolve all aspects of the measurement problem listed, the nature with which it chooses its collapse basis seems *ad hoc* and not ideal, still requiring a yet unidentified underlying mechanism to explain its choice.

By comparison, the environmental decoherence scheme fails to resolve the measurement problem [48, 68, 5] as it does not account for the problem of outcomes. It does, however, have a well defined, and satisfactory mechanism by which it selects a preferred basis. As such it seems natural, following the investigation done here, to suggest an environment as a possible mechanism in collapse models. This would allow the collapse basis of the dynamical reduction to be chosen in accord with einselection which would offer a means for the collapse basis to be chosen by the nature of the system itself. This has issues, however, when one considers that collapse models assume that collapse is a universally continuous effect operating on all systems always regardless of the act of measurement or the presence of the environment.

This similarity between these two processes can be seen most apparently at the level of the density matrix. There the approximate diagonalization due to the environmental coupling in decoherence and due to the stochastic modifications in collapse models are formally very similar in the case of environmental scattering. In fact, with the collapse constants appropriately chosen, the master equations describing both become mathematically identical. This similarity was pointed out by Joos [84] who used it to question the need for an explicit reduction mechanism for decoherence to resolve the problem of outcomes. Ghirardi, Rimini, and Weber [75], in response to Joos, stated that each formalism results in what should be interpreted as different types of density matrices of the system, regardless of the formal similarity. Decoherence describes an improper matrix that only represents the possible outcomes of local measurements on the system, while collapse theories describe an approximately proper matrix with the state having actually reduced to one of the possible outcomes but the observer not knowing which in a case of classical probability. These conceptual, interpretive, and formal differences must be distinguished when discussing these programmes.

These distinctions, however, are merely a result of the particular manner each program is formulated. The improper nature of the density matrix that results from the decoherence scheme is only so due to the use of a partial trace to derive it. This line has been pursued in [119, 1, 28] where it is found that effective reduction due to weak measurement by the environment does indeed occur.

It has also been submitted that other proposals that are suggested to induce effective reduction in collapse models might also be considered as alternative causes of decoherence. It is hoped that both might result from some alternative underlying universal mechanism, such as interactions with some alternate universal "environment" or other degrees of freedom. Proposals have been made that some description of gravity might full this role [97, 52, 102, 117].

5.2 Interpreting Measurement

In investigating these programs, it is unavoidable that the subject of the interpretations of Quantum mechanics might arise. Most interpretations come about as a result of unresolved

phenomenon like the measurement problem and are designed as attempts to resolve such issues by changing how the underlying theory is interpreted. While it is debatable whether modifications to the underlying mechanics, such as those of collapse theories, could be considered an interpretation, Decoherence is completely interpretation agnostic.

It is that characteristic, along with its status as an experimentally measured and confirmed phenomenon and its resolution of nearly all aspects of the measurement problem baring the problem of outcomes, that give it a special place in the quantum to classical transition. Many an interpretation resort to decoherence to resolve issues in their construction, not unlike what was mentioned for collapse modals above, and in so doing cause discussions of these issues to tend towards philosophy rather than any practical physics. One need only interpret the probabilistic nature with which measurement outcomes are selected. For the majority of those that study physics, that is enough, and the question of interpretations is not currently influential in their work. For the rest, there is a lively debate to engage in.

5.2.1 Alternative Interpretations of Quantum Mechanics

Below we shall briefly summarize a few interpretations as well as an alternate model of modification to standard quantum mechanics, namely non-commuting models, and note the cases where and how decoherence has been invoked. We shall also briefly apply the interpretation to the measurement problem as described above in section 2.3 and describe its approach to each aspect.

The Ensemble interpretation

The Ensemble interpretation, also referred to as the Statistical Interpretation, attempts to assume as little as possible about the standard mathematical formalism, taking Born's probabilistic interpretations of the state vector to its logical extreme. The state vector is interpreted not to represent all available information about the state of the system, but instead can only represent a conceptually infinite *ensemble* of identically prepared systems. In this way, the state vector holds only information about the preparation procedure of a system of interest and does not completely describe any individual system. It is taken that any individual system exists as the classical idea of a particle in an ensemble of identically prepared systems

with its observable perfectly realized with relative frequency given by Born's Rule, $|\langle o|\phi\rangle|^2$ for some observable $\hat{O}$ with eigenvector $|o\rangle$ [13, 12].

Here Heisenberg's uncertainty relation, as well as by extension the inability to measure non-commuting observables, is interpreted as a limitation upon the act of measurement rather than any description of the physical characteristics of the system itself. This is argued via a single slit diffraction experiment where the registration of the final position upon the detector screen also measures the component of momentum parallel to the screen, in contrast to the uncertainty relation. It is only after repeated measurements, each populating an ensemble, that the statistical deviations of the ensemble as a whole satisfy the uncertainty relation [13]. Measurement here is assumed to be equivalent to the preparation of the system. In this way, it can reproduce the collapse into the final system after the traditional idea of collapse.

The Ensemble interpretation indeed resolves, by assumption, the problem of outcomes and final state Objectivity. Its minimalist departure from the formalism results in it offering no preferred basis with which to resolve the Basis ambiguity, though its association of the non-commuting nature of observables with the ensemble instead of any individual system means quantum contextually is lost at the level of the individual. The interpretation of superposition of outcomes as probabilistic alternative options implies that while macroscopic objects would not be found in a superposition, the dispersion of their probability density might result in them being in different locations each time one looks. This to me sounds like a failure at resolving the problem of macro-objectification as well. It is easy to see how the addition of decoherence to the description of the ensemble interpretation would allow its ability to resolve these issues to be applied, making this interpretation possibly close to a true resolution of the measurement problem, if perhaps incomplete.

As is typical for an epistemic interpretation, the ensemble interpretation fails to explain any dynamical means by which a particular outcome might be arrived at. While this is not an expressly hidden-variable theory, it is assumed that some yet to be identified mechanism exists to describe the dynamics of the individual systems. It is in this way that the ensemble interpretation and Bohmian mechanics, described below, overlap. Regardless of the exact mechanism, it is assumed that the addition of extra degrees of freedom is required to completely describe the state of an individual system [106].

Consistent Histories

The consistent histories interpretation is best understood from the perspective in which it was first developed. Its original intention was not to overcome conceptual issues within the formalism of quantum mechanics, but rather as a means of describing quantum phenomenon without invoking any concept of an observer, and by extension, the idea of measurement. In so doing, the hope is that descriptions of quantum mechanics in terms of classical properties can be had in a consistent manner [79, 80].

It was initially developed to describe the evolution of an undivided closed system, typically that of the universe, for use in the fields of quantum cosmology. This description is done with the mathematical tools of traditional quantum mechanics, but replaces any idea of collapse with the study of quantum histories defined as a time-ordered sequence of events represented by projection operators that act on the system. The set of all possible histories of a particular system is constructed with each being prescribed a probability via the co-called decoherence functional, which is the natural extension of the existing rules for prescribing probabilities for single events.

In order to achieve the hope of being able to apply classical logic to statements made about the properties of quantum systems [79], the sets of histories are reduced to subsets of consistent histories, referred to as families. The consistency of two histories is defined in terms the consistency condition, which is given in terms of the decoherence functional, and ensures that the probabilities assigned to histories be classical in nature, in that the probability assigned to the history that results from the combination of two histories within the same family is given by the addition of their respective probabilities. As a result of this construction, logical statements can be made that relate histories within a given family [80, 79, 108].

The consistency condition demands the absence of interference between two consistent histories, relating the condition with the emergence of classicality. However it was shown that most consistent histories are rather non-classical [96, 56, 108]. As a result, the interpretation can be modified to describe open systems with the introduction of decoherence, hence the alternate name of the interpretation of decoherent histories. The introduction of decoherence

into the interpretation allows for the selection of stable quasiclassical histories via einselection of a preferred set of "pointer projectors" [96]. This then also gives a physical motivation for the consistency condition.

Within this interpretation, the act of measurement is relegated to an event, along with any other interaction, on that system's history. As such, measurement events appear as a projection on the state, resolving any issues of final state objectivity and fixing the outcome. The issue of macro-objectivity and basis ambiguity is then only resolved after the introduction of decoherence, as was mentioned above.

As this is once again an epistemic interpretation the question of how a particular history is chosen by the system is still an open question. This is relegated to the role of some undescribed degree of freedom, whether hidden-variables or some external influence. In fact, the consistent histories construction allows for only a probabilistic description of the system between events, given a particular choice of history. While it is noted that the original intention of the interpretation was not intended to resolve any measurement problem, it is none-the-less often applied to the quantum-classical transition.

It is interesting to note the conceptual similarities between the consistent histories interpretation and Feynman's path integral formalism of quantum mechanics, with the Feynman paths being interpreted as a family of consistent histories. It has also been noted that the collapse operators in the context of quantum state diffusion have been found to define an approximately consistent history [54].

Bohmian mechanics

Also referred to as de Broglie–Bohm theory, this modification on de Broglie's pilot wave theory first proposed by Bohm [32, 33], posits the existence of physical classical particles with definite positions along with the wave function whose evolution is given by the standard Schrödinger equation. The dynamics of the particle positions is then given by a "guiding equation" which is governed by the underlying wave function resulting in the particles in the system follow deterministic trajectories. Any description of the state of the system is then extended to include the positions of the particles, however, these positions are assumed to be initially hidden with their distribution given by the quantum equilibrium distribution which

is this theories version of Born's Rule. Bohmian mechanics then recreates any prediction about the positions of particles that might come from standard non-relativistic quantum mechanics [108, 22, 91].

This fundamental dependence on the position of the particles makes Bohmian mechanics a contextual theory with position being the only property that truly "exists" for the particle [42]. Any other property, such as spin, is always measured in the context of the final position of the particle making position the only property that is required. In Bell's words [27]

> "In physics the only observations we must consider are position observations, if only the positions of instrument pointers."

While the choice of position as the fundamental physical property is not chosen arbitrarily, its choice is based on the experience of the experimenter rather than any physical phenomenon. It is, however, possible to construct equivalent theories for other choices of fundamental properties, however this would leave position as a contextual property making it difficult to build a coherent description of natural phenomenon. Contextuality in any hidden-variable theory is unavoidable in order to be consistent with quantum mechanics [86, 88], along with the required non-locality due to Bell's theorem.

It is this non-locality within Bohmian mechanics that is restricting any attempts of reconciling it with relativity. While the non-locality of this theory allows it to bypass Bell's Theorem, with the motion of each particle in a system being determined instantaneously by the positions of every other particle, it is a rather flagrant violation of special relativity in which simultaneity is dependent on one's choice of coordinates.

A further criticism of the theory is that there are in principle infinite theories similar to Bohmian mechanics, differing only in their assignment of the particle trajectories [43], calling into question the particular trajectories described by Bohmian mechanics [22]. Additionally, trajectories chosen by this theory are highly non-classical [10] making it difficult for the theory to account for the trajectories of macroscopic objects. Attempts have been made to introduce decoherence into the theory to account for the non-classical nature of the trajectories, as well as give some physical motivation for the particular choice of position as the fundamental variable, but the approach still has many issues to be resolved [108].

Regardless of these issues, Bohmian mechanics is still one of the few theories that offer a means of resolving the measurement problem in quantum mechanics, irrespective of the non-classical nature of its macroscopic trajectories and built-in preference for a particular spacial basis. The underlying guiding wavefunction explains all wave-like phenomenon such as interference, while the particles give a reason for the exact outcome of a measurement, resolving the problem of outcomes. The classical nature of the particles also accounts for macro-objectification and final state objectivity.

It has already been pointed out that Bohmian mechanics can be interpreted as a Hidden-Variable extension of the ensemble interpretation, but we also draw attention to the conceptual similarity of Bohmian mechanics with the consistent histories scheme where the trajectory here is a history under constant spacial projection. This similarity is generalized in the modal approaches to interpreting quantum mechanics, described below.

The Modal Approach to Quantum Interpretations

Modal interpretations refer not to any single interpretation, but rather a class of interpretations with similar properties. The core properties with which to identify the modal approach to interpretations is that of realism achieved via the breaking of the eigenvalue-eigenstate link. Here realism refers to the identification of reality as something that exists in the sense that systems have definite properties, though the particular choice of properties might change over time. In this way the approach allows for the identification of definite measurement outcomes without any concept of wavefunction collapse by disregarding if the state of the system is in an eigenstate of the observable or not, preserving unitary evolution [108, 22, 49, 47, 46].

The common means by which this is achieved is usually by the identification of the system with two separate states: the dynamical state, which describes the possibilities of the state in a probabilistic way, and the value state, which is then the actual state of the system. The dynamical state is normally associated with the wave function of traditional quantum mechanics and is here interpreted as describing the probabilities with which the system might in the future obtain, or could have in the past possessed certain values. The value state represents the true physical properties of the system, each sharply defined at any given instant [49, 46]. As such, interpretations of this kind are often, but not always, considered

Hidden-Variable theories as the value state is often outside the observers reach. There are some that merely adjust the semantics of quantum mechanics [105], avoiding being associated with any hidden-variables. It is in the assigning of the particular properties to be well defined at any point that various interpretations within the modal approach differ. In fact, as already discussed, Bohmian mechanics is such a modal interpretation where the physical property assigned to the value state is fixed as spacial position [40, 35]. There have also been attempts to reconcile the theory of decoherent histories with the modal approach where the histories are treated as the value state equivalent [45, 81]. Environmental decoherence has also been invoked as a possible mechanism for selecting the properties that the value state is defined in [108].

If the goals of the modal interpretations [40] are reached in a consistent manner, the theory would automatically bypass the measurement problem, as is intended by its proponents. However, it would seem that no such ideal modal theory has yet been proposed. Each has various issues in motivating and verifying their respective methods of property assignment [40, 49]. One such issue that makes resolving basis ambiguity an issue is the property assignment might be contextual to the manner in which one might subdivide a complex system [71, 121, 40] such as that of an apparatus interacting with a measured system of interest.

Non-Commuting Quantum Mechanics

While non-commuting quantum mechanics is not often studied in the context of the measurement problem, the work by Adler [3, 4] on what has been denoted Trace dynamics means that its formulation beyond this one theory is worth considering here. It is most commonly associated with the search for a consistent theory of quantum gravity, where the idea to introduce non-commutativity between position operators follows from the theory that localizing systems to scales shorter than the Planck length might cause gravitational instabilities [55]. Alternatively, Adler argues that the current method of canonical quantization for deriving standard quantum theory, which is supposed to be the more fundamental theory, is circular, requiring the knowledge of the final algebra [4].

Regardless of the chosen motivation, the broad approach to the formalism is to identify some underlying configuration space analogous to the phase space coordinate system that

underlies traditional quantum mechanics [113]. For constant commutation and "fuzzy space" theories this takes the form of a one- or two-mode Fock space respectively [112]. For the case of trace dynamics, it's the vector space of infinite Grassmann matrices with an inner-product defined as the trace in the normal way [4, 26]. A selection of operators on this underlying configuration space are then the "state vectors" used to construct a non-commuting analogue Hilbert space in which quantum mechanics is described. For constant commutation and "fuzzy space" theories this Hilbert space is defined with an inner-product defined as the trace over the configuration space [113]. It should be noted, however, that while the more traditional non-commutative approaches work to modify the fundamental formalism of quantum mechanics while recreating the predictions of standard quantum mechanics [113, 4] in the appropriate limits, trace dynamics work to show how standard quantum mechanics emerge as the statistical thermodynamics of this underlying configuration space [4, 3, 26].

These theories can be considered in their infancy and have various issues still to be overcome, such as the breaking of Lorentz invariance, time-reversal symmetry, and highly non-local Hamiltonian. Regardless, they also suggest alternative interpretations of various phenomenon such as the emergence of spin-like effects [118], and the existence of a maximum energies and speed [112]. Currently, any observable differences caused by non-commuting modifications to quantum mechanics are out of experimental reach [112, 69], but the fact that these are testable theories is promising.

As such, we speculate about the possibility of such theories contributing to the discussion on the measurement problem. Trace dynamics have been associated with collapse theories where Brownian motion fluctuations cause modifications to the emergent Schrödinger equation that are similar to those of the CSL model [26, 4]. Identification of the configuration space as an extra degree of freedom leads to recognizing spacial states in the non-commutative description as having intrinsic internal structure. No such investigations into the results that might follow from this observation have been considered, leaving this as a possible future investigation.

5.2.2 Invoking Decoherence

With the inclusion of decoherence into the descriptions of quantum mechanics, the measurement problem is only that of the problem of outcomes, avoiding any contradictory results, giving the role of the environment a seemingly rather important place in the description of the quantum to classical transition.

Is it sufficient then for an interpretation to account for only the singular outcome during measurement for the problem of measurement to be considered resolved? If this is the case, then is it not sufficient to take the position of the Ensemble interpretation with the addition of decoherence and state that the system merely is in a particular eigenstate always, regardless of the state-vector, essentially doing away with the eigenstate-eigenvalue link? This was certainly the position of Einstein for a time [12] and, I speculate, the unintentional means by which many might interpret quantum mechanics in practice, regardless of whether they formally might take the position of the Copenhagen, or some other, interpretation. If the first question is instead responded with the negative, then, as it stands, it would seem that our description of quantum dynamics requires additional parts to account for how a superposition state evolves such that a single outcome is recorded. The current options here are then some sort of dynamical collapse model or hidden-variable theories. Both of which are having troubles in the relativistic domain.

It is only prudent that the alternative category of interpretations is acknowledged. They being what has been dubbed participatory realist interpretations [37], though generally associated with informational interpretations, including the Relational interpretation, Quantum Bayesianism, and perhaps quantum Darwinism described above. Here the probabilities of the measurement process are interpreted as only pertaining to measurement outcomes and the experience of an observer in a real objective reality. In this way, these interpretations avoid invoking decoherence entirely. The wave function is a representation of the knowledge or belief an observer has of a system of interest which updates once new information is gained via measurement. This, however, implies that there is information about the system that we don't know and cannot access, other degrees of freedom outside our reach.

Then there is Everett's Many-Worlds interpretation [62]. Here quantum mechanics is ap-

plied unaltered with the observer becoming another state in the von Neumann chain. Each superimposed chain is then considered alternate branches, each describing a version of reality in which a particular outcome was found. In order to suppress interference between these branches, decoherence is once again used. In this way, the many-worlds interpretation is completely deterministic with reality "splitting" into alternate branches whenever a measurement is performed. Probability then enters due to an observers lack of complete knowledge of the state of the system given their position within the von Neumann chain. This definition however only makes sense within the context of a single branch since once a measurement is complete you find your apparatus to be in a particular outcome state with probability one. Across more than one such branch, the probability of a particular measurement outcome no longer makes sense as you would find the probability to be greater than one. As such we can only make a statement about probabilities in the context of a single branch, but this choosing of a single branch is equivalent to the original collapse postulate, leaving the interpretation without any resolution to the measurement problem [122]. Further, the tail problem applied to environmental decoherence causes interpretational issues for the Many-Worlds interpretation for the same reasons as it does for Collapse models [117].

The role of decoherence in the quantum-to-classical transition would seem to be rather ubiquitous, with only non-commutative theories and collapse models of those discussed not invoking it. Informational interpretations happen to avoid it, but still acknowledge the presence of unknown or unobtainable information within the underlying objective reality. It is then noted that the nature of the decoherence scheme is rather dependent on the role of the environment as an information bath, or sink, into which information can be sent and become inaccessible. This theme of missing or inaccessible information or mechanism seems prevalent in any realist approach to the measurement problem.

5.3 Extra Degrees of Freedom

It appears the majority of interpretations invoke, perhaps not environmental decoherence *per se*, but some extra degree of freedom, whether internal, as is explicitly the case of hidden-variable theories, or external, such as the environmental bath in decoherence or the unknown

mechanism alluded to in collapse theories.

The identification of the role of extra degrees of freedom in the quantum to classical transition is in no way surprising, though it seems that this similarity in the various approaches to the problem is not often stated. Breaking linearity or unitarity, as was described in section 2.4, or the eigenvalue-eigenstate as suggested by the Modal interpretations, while still staying true to standard quantum mechanics in the appropriate limit, naturally invites extra degrees of freedom into the descriptions. As the dynamics transition from quantum to classical, something must emerge as to change its description, whether the interactions with the environment or some yet identified mechanism.

In recognizing the role of extra degrees of freedom in interpretations of quantum mechanics, one cannot miss the relationship with hidden-variable theories. They were, after all, suggested in a similar context. While it must be emphasised that the idea of extra degrees of freedom discussed here are in no way hidden-variables as they are not necessarily intrinsic to the state of the system, hidden-variables, if they exist, would constitute as extra degrees of freedom, as mentioned. While it might be obvious, it is stated anyway. Any extra degree of freedom that is unobservable and intrinsic to the system would by definition be a hidden-variable.

Since the discussion on hidden-variables is extensive, it won't be added to here. Instead, it is noted that the formalism of decoherence relies only on the existence of a set of extra degrees of freedom that are not under investigation during the process of measurement. Whether these are internal or external seem to be beside the point, only that at an accumulative level their combined states tend to orthogonality. By decoherence here it is meant the approximate diagonalization of the reduced density matrix of a system of interest into an improper mixture. I clarify this point to disconnect the term as used here from other results that follow from the environmental decoherence scheme, such as the resolution of the basis ambiguity, which might not apply since not all sets of extra degrees of freedom can be treated as a bath.

5.3.1 Internal Decoherence

Here, inspired by the decoherence formalism, the role of micro-systems on macroscopic observables in a complex system is investigated. As such we start with a system of N quantum

mechanical micro-systems ϕ_n. For simplicity we assume each micro-systems exists in identical D dimensional Hilbert spaces $\mathcal{H}_n$. We define some Hermitian operator in this space with the eigenvalue equation $\hat{o}_n \left|d\right\rangle_n = o_{d_n} \left|d\right\rangle_n$ where the index $d \in [1, D]$ labels the state and associated eigenvalue. An arbitrary state of a given micro-system $\left|\phi\right\rangle_n \in \mathcal{H}_n$ is then

$$\left|\phi\right\rangle_n = \sum_d^D \phi_n(o_{d_n}) \left|d\right\rangle_n , \tag{5.3.1}$$

with $\phi_n(o_{d_n}) = {}_n\langle d|\phi\rangle_n$.

Each micro-system then interacts with every other micro-system via some interaction Hamiltonian $\hat{H}_I(\hat{\vec{\mathbf{o}}})$ with $\hat{\vec{\mathbf{o}}} = (\hat{o}_1, \hat{o}_2, \cdots, \hat{o}_n, \cdots, \hat{o}_N)$, which in turn is defined on the Hilbert space of the complex macro-system given by $\mathcal{H} = \bigotimes_n^N \mathcal{H}_n$ with dimension $\dim(\mathcal{H}) = D^N$. Its basis vectors are then $\left|\vec{\mathbf{d}}\right\rangle = \bigotimes_n^N \left|d\right\rangle_n$, giving an eigenvalue equation $\hat{\vec{\mathbf{o}}} \left|\vec{\mathbf{d}}\right\rangle = \vec{\mathbf{o}}_{\vec{\mathbf{d}}} \left|\vec{\mathbf{d}}\right\rangle$ where $\vec{\mathbf{o}}_{\vec{\mathbf{d}}} = (o_{d_1}, o_{d_2}, \cdots, o_{d_n}, \cdots, o_{d_N})$.

With this construction in place, we define a macro-observable $\hat{M}$ that acts on the entire complex Hilbert space $\mathcal{H}$ and is associated with some macroscopic quantity such as average magnetization or total energy. Here such an observable is represented by the form

$$\hat{M}(\hat{\vec{\mathbf{o}}}) = \sum_n \omega_n \hat{o}_n = \vec{\omega} \cdot \hat{\vec{\mathbf{o}}}, \tag{5.3.2}$$

where the constants ω_n are fixed by the nature of the observable. It is simple to see that the eigenvalues of our macro-observable are then $\vec{\omega} \cdot \vec{\mathbf{o}}_{\vec{\mathbf{d}}} = m$, and from the commutativity of the macro-observable with the interaction observable, $\left[\hat{M}, \hat{\vec{\mathbf{o}}}\right] = 0$, the basis states are degenerate eigenstates for the macro-observable, $\hat{M} \left|\vec{\mathbf{d}}\right\rangle = m \left|\vec{\mathbf{d}}\right\rangle$. Since it is these macroscopic quantities that one might wish to measure, we would like to label the basis states with its eigenvalue. It is here that we note that the specification of m also uniquely defines M subsets $S_m \subset \mathcal{H}$ of the basis states,

$$S_m = \left\{ \left|\vec{\mathbf{d}}_m\right\rangle \in \mathcal{H} \mid \vec{\omega} \cdot \vec{\mathbf{o}}_{\vec{\mathbf{d}}} = m \quad \text{where } \hat{\vec{\mathbf{o}}} \left|\vec{\mathbf{d}}_m\right\rangle = \vec{\mathbf{o}}_{\vec{\mathbf{d}}} \left|\vec{\mathbf{d}}_m\right\rangle \right\}, \tag{5.3.3}$$

with each being disjoint, $S_m \cap S_{m'} = \emptyset$ and M is the number of unique eigenvalues m of the macro-observor $\hat{M}$. As such, we can label the basis states as $\left|m, \vec{\mathbf{d}}_m\right\rangle$, with $\vec{\mathbf{d}}_m$ labeling different degenerate states with eigenvalue m as to differentiate them.

A generic state in the total complex Hilbert space $\mathcal{H}$ is then given by

$$|\Phi\rangle = \bigotimes_n^N \sum_d^D \phi_n(o_{d_n}) \, |d\rangle_n \tag{5.3.4}$$

$$= \sum_{\vec{\mathbf{d}}}^{D^N} \Phi(\vec{\mathbf{o}}_{\vec{\mathbf{d}}}) \left|\vec{\mathbf{d}}\right\rangle \tag{5.3.5}$$

$$= \sum_m^M \sum_{\{S_m\}} \Phi(\vec{\mathbf{o}}_{\vec{\mathbf{d}}_m}) \, |m\rangle \left|\vec{\mathbf{d}}_m\right\rangle, \tag{5.3.6}$$

where $\Phi(\vec{\mathbf{o}}_{\vec{\mathbf{d}}}) = \prod_n^N \phi_n(o_{d_n})$. The summation over the vector index $\vec{\mathbf{d}}$ in the second line should be understood to be a summation over the space of N-tuples $\vec{\mathbf{d}} = (d_1, \cdots, d_n, \cdots, d_N)$ with $d_n \in [1, D]$. In the last line we have separated the summation over the basis states into a summation over the subsets S_m and the possible eigenvalues m of the observable $\hat{M}$. In so doing we have done the equivalent of separating a centre of mass co-ordinates and relative co-ordinates by redefining the Hilbert space $\mathcal{H}$ as two coupled sub-spaces $\mathcal{H}_M$ and $\mathcal{H}_S$. Here $\mathcal{H}_M$ is the Hilbert space with basis vectors defined as the eigenvectors of the macro-observable $\hat{M}$, denoted here as $|m\rangle$, and $\mathcal{H}_S$ the Hilbert space of the internal relative states of the system whose basis are denoted $\left|\vec{\mathbf{d}}_m\right\rangle$.

We can now treat these relative states as an "environment" in accord with the decoherence scheme and identify the reduced density matrix of the system in the Macro-observables Hilbert space $\mathcal{H}_M$. Identifying the density matrix ρ of the total system and performing a partial trace over the relative state Hilbert space $\mathcal{H}_S$,

$$\mathrm{Tr}_S \, \rho = \mathrm{Tr}_S \, |\Phi\rangle \langle\Phi'| \tag{5.3.7}$$

$$= \sum_{\left|\vec{\mathbf{d}}_{m''}\right\rangle} \left\langle \vec{\mathbf{d}}_{m''} \middle| \Phi \right\rangle \left\langle \Phi' \middle| \vec{\mathbf{d}}_{m''} \right\rangle \tag{5.3.8}$$

$$= \sum_{\left|\vec{\mathbf{d}}_{m''}\right\rangle} \left\langle \vec{\mathbf{d}}_{m''} \right| \sum_{m,m'}^{M} \sum_{\{S_m\},\{S_{m'}\}} \Phi(\vec{\mathbf{o}}_{\vec{\mathbf{d}}_m}) \Phi^*(\vec{\mathbf{o}}_{\vec{\mathbf{d}}_{m'}}) \left|\vec{\mathbf{d}}_m\right\rangle |m\rangle \langle m'| \left\langle \vec{\mathbf{d}}_{m'} \middle| \vec{\mathbf{d}}_{m''} \right\rangle \tag{5.3.9}$$

$$= \sum_{m,m'}^{M} \sum_{\{S_m\},\{S_{m'}\}} \Phi(\vec{\mathbf{o}}_{\vec{\mathbf{d}}_m}) \Phi^*(\vec{\mathbf{o}}_{\vec{\mathbf{d}}_{m'}}) \, |m\rangle \langle m'| \left\langle \vec{\mathbf{d}}_{m'} \middle| \vec{\mathbf{d}}_m \right\rangle \tag{5.3.10}$$

$$= \sum_m^M \mathcal{P}(m) \, |m\rangle \langle m|, \tag{5.3.11}$$

where $\mathcal{P}(m) = \sum_{\{S_m\}} \left|\Phi(\vec{\mathbf{o}}_{\vec{\mathbf{d}}_m})\right|^2$. The orthogonality of the relative states, $\left\langle \vec{\mathbf{d}}_{m'} \middle| \vec{\mathbf{d}}_m \right\rangle = \delta_{m',m}\delta_{\vec{\mathbf{d}}_{m'},\vec{\mathbf{d}}_{m'}}$, follows from them still being basis vectors for the total complex Hilbert space $\mathcal{H}$, with $\left\{\left|\vec{\mathbf{d}}_m\right\rangle\right\} = \left\{\left|\vec{\mathbf{d}}\right\rangle \mid \vec{\omega}\cdot\vec{\mathbf{o}}_{\vec{\mathbf{d}}} = m\right\}$, as well as each uniquely defining a particular eigenvalue m of $\hat{M}$ via there place in the disjoint set S_m.

Related results, in differing contexts, have been found elsewhere by alternative means [11, 82, 57] and is in no way surprising. It follows from the contextual nature of the decoherence scheme [57, 125, 124], where the distinction of subsystems as system and environment changes the nature of the decoherence. One then applies the Schmidt decomposition theorem [107, 111] to the particular choice of division made by the observer. One then can provide a coarse-grained description of the subsystem chosen as the system of interest. This results in the reduced density matrix of a system associated with some macro-observable $\hat{M}$ being perfectly decohered with the probability of measuring interference perfectly zero. This is in contrast to the minuscule but non-zero interference terms that result from traditional environmental decoherence. Additionally, this result has no dependence on the nature of the Hamiltonian of the total system and emerges without any time evolution.

This process can also be generalized to a continuous basis. Though here it is pointed out that the interpretation of the density matrix as an improper mixture must not be forgotten. The lack of interference terms in no way means that the system has reduced to a proper mixture of pure state density matrices. Instead, it is merely the case that the effects of interference cannot be measured from the macro-observable $\hat{M}$, they are seemingly hidden by the degenerate extra degrees of freedom of the system. It would seem that for certain complex systems, one can define a macro-observables whose measurement outcome space seems classical in nature.

As a final note, a conceptually similar result was found in the context of collapse models in section 4.1.1. There the collapse mechanism was found to only apply to the centre of mass coordinate with the relative coordinates evolving according to the standard Schrödinger equation.

5.4 Summary and Conclusion

In this thesis, we have reviewed the measurement problem as well as two popular programs that are applied to its solutions, namely the Environmental Decoherence Scheme and Dynamical Collapse models. The similarity between each is noted, as well as similarities with various other interpretations of Qunatum mechanics. Here we will briefly summarize each of these chapters.

After defining the postulates of quantum mechanics, the problem of measurement is identified as underlying the lack of consistency between our description of quantum mechanics and our observations of macroscopic phenomenon. This we demonstrate within the context of the double split experiment in which the dichotomy between the behaviour of the particle and its measured outcome state, often referred to as the wave-particle duality, is an illustrative example of this inconsistency. The concept of von Neumann's ideal measurement and the effect of the measuring apparatus on the system are introduced here. Having a sense of the issue at hand, it's aspects are defined in clear terms. These include the problem of Macro-Objectivity, the problem of Outcomes, as well as the lesser (though no less important) issues of the emergence of Born's Rule, Basis Ambiguity, and the objectivity of the final state. We then discuss what might be required to overcome these issues as listed. The possibilities of breaking from the Unitarity, or alternatively, the Linearity of our descriptions of quantum dynamics is suggested.

In the review of Environmental Decoherence, we introduce the idea of weak measurement by an environment that weakly interacts with a system of interest. There it is found when exposed to this process continuously, the interference between alternative superimposed states is suppressed. With this concept in hand, the formalism of the environmental decoherence scheme is described in terms of partial traces over the environmental degrees of freedom of the combined system-environment state. Within this context, decoherence amounts to the suppression of the off-diagonal terms of the density matrix, representing the interference effects in the system. This results in a final system state in an improper mixture, an important interpretive point that differentiates this result from that of a reduced density matrix in a proper mixture. Further results of the decoherence scheme are investigated such as the envi-

ronmental induced selection of a set of preferred basis states, referred to as the pointer states, in which the process of decoherence occurs. This then naturally results in the resolution of issues of basis ambiguity.

The role of the environment in the process of measurement is further increased to include its role as a communication channel by which information about the system propagates and is stored. It is found that the information that is most successfully propagated through the environment, measured by the redundancy with which it is stored, is that information of the system in the basis of the environmentally selected pointer basis. This offers an explanation as to the objective nature with which observers choose their basis of measurement and resolves issues of the objectivity of the final state outcomes. This process, that results from the environment as a witness scheme, is called quantum Darwinism after the analogy of survival of the fittest information successfully propagating through the environment.

We then review the Dynamical reduction program and its approach to the measurement process. There it is augured that the linear dynamics as described by the Schrödinger equation is only accurate as the limit of some other non-linear dynamics. To this end the dynamical reduction scheme opts to modify the Schrödinger equation as to add terms that drive the system towards reduction. The source of these terms are yet to be identified, but the resulting mechanism is designed to include an amplification process such that while reduction for a single system might take on the order of the lifetime of the universe, for a number of systems of the order of Avogadro's number, the resulting reduction is sufficiently fast. In this way, such models manage to simultaneously describe the dynamics of microscopic quantum systems as well as the macroscopic classical systems.

We investigate the three most popular such collapse models, namely the Ghirardi, Rimini, and Weber (GRW), Continuous Spontaneous Localization(CSL), and Quantum Mechanics with Universal Position Localization (QMUPL) models. In the GRW framework reduction occurs as the result of instantaneous localization due to a Localization operator. Such an operator only applies with some small frequency, but when such an event occurs, the system instantaneously localizes to a new state given as the old state multiplied by a Gaussian function. A master equation for this model is given from which it is shown that when the system is separated into the centre of mass and relative co-ordinates, the localization process

only affects the centre of mass variable at an amplified rate, demonstrating the amplification process mentioned above.

The CSL model is then reviewed. This model is the most accurate of the ones listed, but its comparative mathematical complexity makes it unwieldy to use. Here a set of real values Wiener processes are coupled to collapse operators chosen by the experimenter subject to certain conditions. The model is continuous in time with a stochastic differential equation that results from the modification of the Schrödinger equation. The most common choices for the collapse operators are given, namely the local average density operator and the smeared mass density operator, each giving similar results. It is then shown how reduction in the position basis occurs in this model and the emergence of Born's Rule. The rate of reduction is calculated and the amplification mechanism in this model is discussed.

We then move onto the final model, that of the QMUPL model. This model is the continuous limit of the instantaneous GRW model, and as such has its own stochastic differential function, whose form is reminiscent of the CSL model but with the position operators for each particle in the system chosen as collapse operators. Within this framework the statistical dynamics of particular observables are investigated and shown to match what is expected, barring energy which is shown to increase at an undetectable rate.

The lack of specification on their choice of parameters is discussed. This follows from the lack of any underlying mechanic that drives the reduction process. As a result, there is still a fair amount of space in the particular choice of the parameters that describe the reduction process while still matching experimental results. Generalizations into non-white noise and the addition of dissipative effects to combat the lack of conservation of energy are also discussed. Each is shown to still reproduce similar results to the basic unmodified models, except that the addition of dissipative effects causes the energy of the system to reach a fixed energy in the long time limit. This is associated with the thermalization of the system to the surrounding bath. Finally, the tails problem in collapse models is discussed. This amounts to the non-zero, but minuscule interference terms in the proper mixed density matrix of the system that results from the reduction process. It is pointed out that this same problem exists in the Decoherence scheme, though the difference in interpretations of the density matrices makes this a more glaring issue in the case of collapse models.

in the Final discussion, decoherence and collapse models are discussed in regards to their approaches to the measurement problem. It is stated how decoherence fails to resolve the problem while collapse models have no natural way to assign their collapse operators. It is further pointed out that other than the interpretational differences of the final density matrix, both schemes have identical master equations for the case of environmental scattering. The idea that the unknown mechanism driving collapse in the collapse models might share a source with decoherence is suggested.

This idea is extended to various other interpretations of quantum mechanics, a few of which are briefly and broadly described, along with their respective approach to the measurement problem as described above. These include The Ensemble interpretation, Consistent Histories, Bohmian Mechanics, Model Interpretations and finally non-commuting quantum mechanics which is technically not an interpretation but rather a modification. The way in which each has invoked environmental decoherence to resolve flaws in their approaches is discussed, as well the tendency for the interpretations to rely on some unknown or extra degree of freedom.

The formalism of decoherence is then applied to a complex system in which internal degenerate relative degrees of freedom are assigned the role of the environment, while the system of interest is associated with some macroscopic observable. This then is shown to produce a reduced density matrix for the macro-observable system with exactly zero interference terms.

In conclusion: The measurement problem is intimately tied to the fundamental nature of quantum mechanics as a description of reality. It speaks to the border on the quantum-to-classical transition, and as such its resolution is the key to its understanding. Many and varied approaches to this issue have been attempted, the most popular of which is environmental decoherence and dynamical collapse models, prompting their revision here. While Decoherence fails to resolve the problem, it none-the-less plays a pivotal role in its resolution. The manner in which many interpretations turn to it speak to this role in measurement. However, the use of the environment as the chosen degrees of freedom, while fundamental for the results of quantum Darwinism whose role cannot be understated, might miss the mark. Perhaps what degrees of freedom are considered for that particular role should be evaluated, as well as restrictions on such a pursuit. This might be the strength of collapse models as

they find results without specifying the underlying degrees of freedom.

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